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Colloquium Paper

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Complexity, contingency, and criticality

(macroevolution/macroeconomics/punctuated equilibrium)

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ABSTRACT **Complexity originates from the tendency of large dynamical systems to organize themselves into a critical state, with avalanches or "punctuations" of all sizes. In the critical state, events which would otherwise be uncoupled become correlated. The apparent, historical contingency in many sciences, including geology, biology, and economics, finds a natural interpretation as a self-organized critical phenomenon. These ideas are discussed in the context of simple mathematical models of sandpiles and biological evolution. Insights are gained not only from numerical simulations but also from rigorous mathematical analysis.**

I. Introduction

Complexity lies in the details, but science deals with generalities. How, then, can one possibly imagine a science of complexity? The eminent biologist and science writer, Steven Jay Gould, has articulated this paradoxical question in his book "Wonderful Life" (1).

> How should scientists operate when they must try to explain the results of history, those inordinately complex events that can occur but once in detailed glory? Many large domains of nature—cosmology, geology, and evolution among them—must be studied with the tools of history. The appropriate methods focus on narrative, not experiment as usually conceived.

Here, we present a different approach, which invokes the traditional scientific method, rather than narrative, to study complex phenomena. We shall argue that self-organized criticality (SOC) (2–5) underlies the widespread appearance of contingency and complexity in nature. In particular, the statistics of large-scale behavior obeys fundamental laws of nature, even though the individual events themselves are unique. These fundamental laws are "universal" and describe many different types of systems. We will apply this analysis to two domains referred to by Gould—namely, geology and evolution—as well as macroeconomics. This approach asks different types of questions than the narrative method and leads to a complementary way of understanding complexity in nature. This is achieved by rigorous mathematical analysis, as well as numerical studies, of simple models. Before going into the details of our method, we will explore, in general terms, what a science of complexity could be.

History Versus Science. Traditionally, sciences may be grouped into two categories: "hard" sciences, where repeatable events can be predicted from a mathematical formalism expressing the laws of nature, and "soft" sciences of complex systems, where only a narrative account of distinguishable events, in hindsight, is possible. Physics is simple but relatively difficult; history is complicated but perhaps easier. Historical events depend on freak accidents, so that if the tape of history were replayed many times, the outcome would differ vastly each time. The mysterious occurrence of incidents leading to dramatic outcomes has fascinated historians as well as fiction writers. Would the Second World War have occurred if Hitler had not been born? Historians explain events in a narrative language where event A leads to event B because event C previously had led to events D and E. But suppose that event C did not happen; then, the course of history would have changed into another series of events. These would have been equally well explainable, in hindsight, with a different narrative. History (including that of evolution or geology) depends on contingency. There is nothing wrong with this way of doing science, where the goal is an accurate, narrative account of specific events.

Similarly, as explained by Gould (1), biological evolution depends on one accidental event after another. If the tape of biological evolution were to be replayed millions of times, not once would we see a life form vaguely similar to ours. Contingency rules in the soft sciences, where the concept of detailed predictability is utterly irrelevant. Thus, the goal of a science of evolutionary biology cannot be to explain why we have elephants or humans. Life as we see it today is just one very unlikely outcome among myriads of other equally unlikely possibilities. For example, life on earth would probably be totally different if the dinosaurs had not become extinct, perhaps as a consequence of a meteor hitting the earth instead of continuing its benign periodic orbit.

Similar considerations apply to economics, where large catastrophes may depend on subtle decisions. For example, Black Tuesday in 1987, when the stock market crashed, may be attributed to the introduction of programmed trading. The economist Brian Arthur has beautifully explained how the dominance of one product over another, such as the VHS system versus the Betamax system for video recording, depends on minor accidental events (6). These events are unrelated to the ultimate performance of the competing systems. If the tape of technological innovation were to be shown again, the resulting technology might well be different. Contingency, again, is the key issue.

A Nonequilibrium Approach. What underlying properties of history, biology, and economics make them sensitive to contingency? In other words, what common features lead to interdependence and complexity? Why can accidents occur that have dramatic global consequences? These questions have been raised rarely; the narrative account has been considered to be sufficient explanation.

From a physicist's viewpoint, though, biology, history, and economics can be viewed as dynamical systems. Each system consists of many individual parts that interact with each other. In economics there are many agents, such as consumers,

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Abbreviation: SOC, self-organized criticality.

producers, governments, thieves, and economists. These agents each make decisions optimizing their own idiosyncratic goals. The actions of one agent affect other agents. In biology, individual organisms—or from a more general perspective, individual species—interact with one another. The actions of one organism affect the survivability, or fitness, of others. If one species changes by mutation to improve its own fitness, other species in the ecology are also affected.

In the past, it has usually been tacitly assumed that these types of large systems are in stable equilibrium. The leading economic theory up to now, "general equilibrium theory," assumes that perfect markets, perfect rationality, and so on, bring economic systems into stable "Nash" equilibria. In the equilibrium state, small perturbations, or shocks, will cause only small disturbances that eventually dissipate. In this theory, the system's response to a small impact is proportional to the size of the impact; equilibrium systems are linear. Large fluctuations in equilibrium, noisy, systems can occur only if many random events accidentally pull in the same direction. This type of accidental concurrence is exponentially unlikely. As a result, the distribution of fluctuations in equilibrium systems is Gaussian; large events are sharply cut off.

The general equilibrium paradigm seems to us to be a deeply flawed picture of economics. The general equilibrium theory has not been explicitly formulated for biology, but the picture of nature as being in "balance" prevails in many circles. Nature is something that can, in principle, be conserved; this idea motivates environmentalists and conservationists. No wonder—in a human lifetime very little changes, so equilibrium concepts may seem natural or intuitive. As pointed out by Gould and Eldredge (7, 8), though, this apparent equilibrium is only a period of relative tranquility, or stasis, between intermittent bursts of activity and volatility.

Although we disagree with Gould's view that the traditional scientific method is not appropriate to study complex phenomena (1), we completely agree with Gould and Eldredge's picture of "punctuated equilibrium" (7, 8). We go further, and propose that physical theories of nonequilibrium behavior in evolution and economics, for example, can be constructed. Von Neumann once referred to the theory of nonequilibrium systems as the "theory of non-elephants." Nevertheless, we shall attempt such a theory of non-elephants.

SOC. The basic idea is that large dynamical systems naturally evolve, or self-organize, into a highly interactive, critical state where a minor perturbation may lead to events, called avalanches, of all sizes (2–5). The system exhibits punctuated equilibrium behavior, where periods of stasis are interrupted by intermittent bursts of activity. Since these systems are noisy, the actual events cannot be predicted; however, the statistical distribution of these events is predictable. Thus, if the tape of history were to be rerun, with slightly different random noise, the resulting outcome would be completely different. Some large catastrophic events would be avoided, but others would inevitably occur. No "quick-fix" solution can stabilize the system and prevent fluctuations. If this picture is correct for the real world, then we must accept fluctuations and change as inevitable. They are intrinsic to the dynamics of biology, history, and economics. For the same reasons, we also abandon any idea of detailed predictability. In economics, the best we can do, from a selfish point of view, is to shift disasters to our neighbors. Large, catastrophic events occur as a consequence of the same dynamics that produces small, ordinary events. This observation runs counter to the usual way of thinking about large events.

Empirically, it was pointed out by Mandelbrot (9–11) that, for many systems, the statistical probability distribution of large events is given by the same distribution functions as small events, implying a common dynamical origin. These distribution functions are known as Pareto–Levy functions. The probability of large events is given by the tails of those distributions, which fall off as power laws—i.e., much slower than Gaussian distributions. Mandelbrot (12) coined the term "fractal" to describe scale-free, or power-law, behavior, although he did not consider the physical origins of these fractals. Nevertheless, scientists and others studying the fluctuations of markets would consistently discard large events as anomalous, since each could be attributed to specific "abnormal" circumstances. Contingency was used as an argument for statistical exclusion. Once the large events are discarded, the remaining events trivially obey Gaussian statistics, and the general equilibrium theory can be preserved. As Mandelbrot pointed out, though, this is like throwing out the baby with the bathwater!

A similar blind spot is found in the field of geology. When studying earthquakes, scientists almost universally look for specific mechanisms for large events—again using a narrative, historical description for each earthquake in isolation. This occurs despite the fact that earthquakes follow a glaringly simple distribution function known as the Gutenberg–Richter law (13). The Gutenberg–Richter law, as shown in Fig. 1, is a power law for the probability distribution of earthquakes. The probability to have an earthquake of energy E, $P(E)$, is proportional to E^{-b}, where b is a characteristic exponent. When plotted on a log–log scale, such power laws appear as straight lines where the slope is $-b$. For comparison, a Gaussian distribution is also plotted in Fig. 1, and the sharp cutoff in the tail of the distribution is apparent. The power law has no cutoff; it is a signature of scale-free, fractal phenomena.

Of course, the statistics of the few large events in the tail of the earthquake distribution is obviously poor, leading to a superficial justification for a different treatment of these events. Similarly, Raup (15, 16) has pointed out that the distribution of extinction events in biology follows a smooth distribution where large events, such as the Cretaceous extinction of dinosaurs, occur with fairly well-defined probability. Finally, the occurrence of large-scale structure in the distribution of galaxies in the universe has been taken as an argument against a statistical description of this distribution in terms of fractals (17). On the contrary, we shall see that inhomogeneities occur at any scale, including the largest, for complex dynamical systems. In our view, the present universe is one of many possible outcomes that would emerge if the tape of the history of the universe were to be run again and again. Somewhat counterintuitively, the fact that one can come up with specific narrative explanations for large events does not preclude the possibility that the statistics of these events follow regular laws of nature.

The canonical model of SOC is a sandpile onto which sand is dropped randomly (2–5). The sandpile model will be pre-

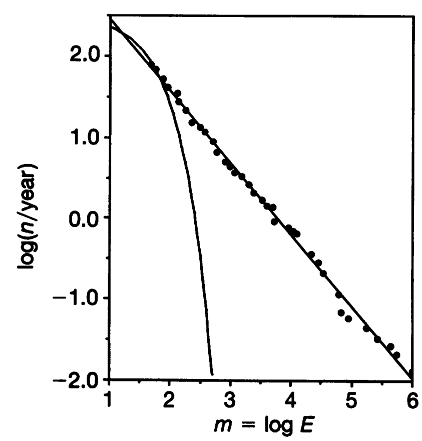

FIG. 1. Distribution of earthquakes in the New Madrid zone in the southeastern United States during the period 1974–1983, collected by Johnson and Nava (14). This power-law, scale-free behavior is compared to a Gaussian curve, which has a sharp cutoff.

sented and discussed in Section II. This model can also be viewed as a model of earthquakes, but more appropriate models of this and other geophysical and astrophysical phenomena have been constructed. In Section III, a simple model of biological evolution (18) will be presented and discussed. With a change of language, this model can be thought of as representing an economy of interacting, evolving agents. Much progress has come from numerical simulations of these and other models, but the model of biological evolution has the added advantage that it is amenable to analytical studies, as discussed in Section IV. In collaboration with Sergei Maslov, we have derived explicit formulae for the self-organization process (19) and for the statistical properties of the resulting critical state (20, 21). Progress from this model has been extended to other self-organized critical models representing growth phenomena (20–23). We believe that it provides a general phenomenology for dealing with contingency and complexity in nature.

II. The Sandpile Paradigm

Can there be a theory of contingency in complex systems? In 1987 one of us, together with Chao Tang and Kurt Wiesenfeld, constructed a model which has become the paradigm of self-organized critical behavior. The model represents the following situation. Consider a pile of sand on a table, where sand is added slowly, starting from a flat configuration. This is a dynamical system with many interacting degrees of freedom, represented by the grains of sand. The flat state represents the general equilibrium state; this state has the lowest energy. Initially, the grains of sand will stay more or less where they land. Eventually, the pile becomes steeper, and small avalanches, or sandslides, occur. The addition of a single grain of sand can cause a local disturbance, but nothing dramatic happens. Eventually, the system reaches a statistically stationary state, where the amount of sand added is balanced, on average, by the amount of sand leaving the system along the edges of the table. In this stationary state, there are avalanches of all sizes, up to the size of the entire system.

The collection of grains of sand has been transformed from one where the individual grains follow their own independent dynamics, to one where the dynamics is global. In the stationary state, there is one complex system, the sandpile, rather than many separate simple grains of sand. Note that in the intermediate state this is not the case. A simple change of the position where sand is added results only in small, local changes to the configuration. The response, a small avalanche, is proportional to the impact. Contingency is irrelevant. Shift a couple of grains left or right, and the resulting state is only marginally affected. The flat sandpile (general equilibrium) or the shallow sandpile does not describe the remarkable, seemingly accidental, occurrence of events; near equilibrium, the outcome is not contingent on specific minor details.

In the resulting stationary state, though, the situation is entirely different. A single grain of sand might cause an avalanche involving the entire pile. A small change in the configuration might cause what would otherwise be an insignificant event to become a catastrophe. Suppose that at some point in time there happens to be a large avalanche, causing devastating destruction to the pile. How would the historian describe what has happened, and how would the physicist? Let us first hear the historian's account of the event.

A Historian Describes a Sandslide. "On December 16, 1994, a grain of sand landed at the site with coordinates [14, 17] on the pile. Adding to the grains of sand already accumulated at this site, this addition caused a toppling of that site, spilling over to the neighboring sites. Unfortunately, one of these sites [14, 18] happened to be near an instability so that the toppling caused this site to topple also. This toppling destabilized sites [14, 19] and [15, 18] and eventually led to the collapse of a large part of the pile.

"Clearly, the event was contingent on several factors. First, had the initial grain of sand fallen elsewhere, nothing dramatic would have happened. Also, if the configuration at position [14, 19] had been slightly different, the sandslide would have stopped sooner, without devastating consequences. While we can give an accurate and complete account of what actually happened, we are at a loss to explain how these many accidental features could possibly have conspired to produce an event of such magnitude. The event was contingent upon many separate, freak occurrences and could clearly have been prevented. Furthermore, we are baffled by the fact that even though sand had been added to the system for a long time, only minor events had occurred before the devastating collapse, and we had every right to expect the system to be stable. Clearly, the event was a freak one caused by very unusual and unfortunate circumstances in an otherwise stable system that appeared to be in balance. Precautions should and could be taken to prevent such events in the future."

The physicist now would give a much more boring and prosaic account of what happened.

A Physicist Describes a Sandslide. "During a long transient period, the pile evolved to a critical state with avalanches of all sizes. We were able to make a rough identification of the toppling rule and to construct a computer model of the phenomenon. Actually, the particular rule that we use is not very important. In any case, we do not have sufficient information about the details of the system to be able to make long-term predictions.

"Nevertheless, our model exhibits some general features of the sandpile. We monitored how many avalanches of each size occurred, after the addition of a single grain to the pile. We made a histogram (Fig. 2), and found that the distribution of events where a total of s sites topple obeys a power law, $P(s) \sim s^{-\tau}$. Thus, if one waits long enough, one is bound to see events that are as large as one has the patience to wait for. We ran our simulations (the tape of evolution) several times. Eliminating the particular grain of sand that caused a particular avalanche only made the system produce large avalanches somewhere else at different times. Changing the rules slightly—for instance, by planting snow screens here and there—does not have any effect on the general pattern. Avalanches are an unavoidable and intrinsic part of the sandpile dynamics.

"Actually, I'm not interested in the specific details of the event which Prof. Historian is so excited about and gives such a vivid account of. What the professor sees as a string of freak events appearing accidentally and mysteriously by an apparent

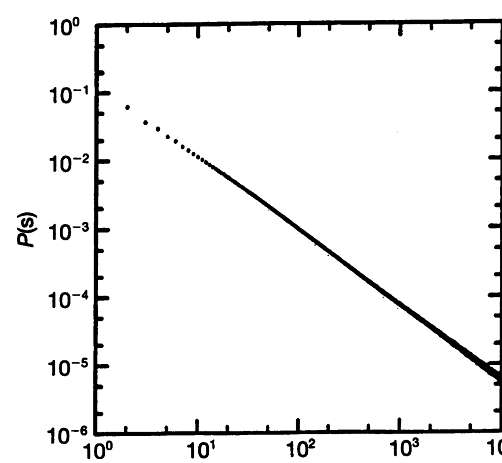

FIG. 2. Power law distribution for avalanches in the sandpile model. Power laws appear as straight lines in double logarithmic plots.

'act of God' and leading to a catastrophe is simply a manifestation of the criticality of the system. History has prepared the sandpile in a state that is far from equilibrium, and the matrix through which the avalanche propagates is predisposed to accommodate events of large sizes. The complex dynamics which is observed in the 'historical science,' where the outcome appears contingent on many different, specific events, represents the dynamics of self-organized critical systems."

Universality. Incidentally, the historian is able to make short-term predictions by carefully identifying the rules and monitoring his local environment. If he sees an avalanche coming he can predict when it will hit with some degree of accuracy. His situation is similar to that of the weatherman: by experience and data collection he can make forecasts, but this gives him no insight into how the climate works. For instance, he does not understand the statistical fluctuations of sunshine, rain, etc.

A computer sandpile model, like the one mentioned above, can easily be constructed. The underlying philosophy is that general features, like the appearance of large catastrophes, and perhaps critical exponents, are not sensitive to the details of the model. Since we understand so little about these kinds of systems, we allow ourselves to study the simplest models that could possibly represent the phenomena under consideration. One guiding principle of physics is the concept of universality. What this means is that important features of large-scale phenomena are grossly insensitive to the particular details of the models and are shared between seemingly disparate kinds of systems. This concept of universality has served well in the study of equilibrium phenomena; in particular, it has been crucial to understanding phase transitions. Of course, we have to demonstrate that our models are robust and insensitive to changes in the rules. If, unfortunately, it turns out that they are not, we are back to the messy situation where detailed engineering-type models of the horrendously complex phenomena is the only possible approach (i.e., the weatherman's approach), and the field is no place for a physicist to be.

The Sandpile Model. The sandpile model is defined as follows. The height of the pile at the point with coordinates (i, j) is denoted $Z(i, j)$. Each height has an integer value. At each time step, the height at some random point is increased by unity, $Z \rightarrow Z + 1$. If the height at that site now exceeds an arbitrary critical height, Z_{cr}, then a toppling event occurs, where the height of the unstable site is reduced by 4 units and the height at each of the four neighbors on the square lattice is increased by 1 unit. This is true except along the boundaries where sand is thrown out of the pile. If any of the neighboring sites are now unstable ($Z > Z_{cr}$), the process continues until none of the Z values in the system exceeds the critical value. Then the avalanche is over, and a new avalanche can be started by adding another grain of sand to the system. The total number of topplings during the avalanche is counted; this

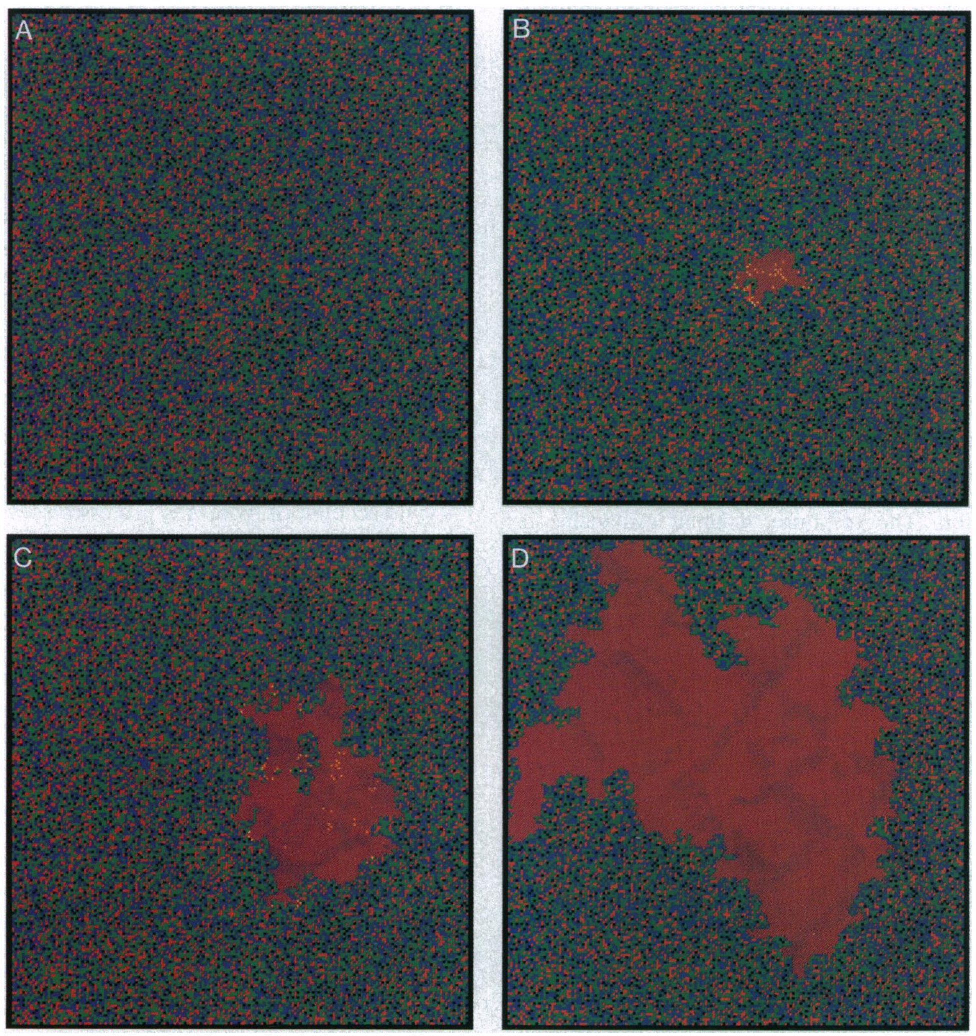

FIG. 3. Evolving avalanche in sandpile model. (*A*) The configuration before a grain of sand is dropped. The various colors indicate heights 0–3, with 3 being the critical height. (*B–D*) Snapshots during the avalanche. The red color indicates sites that have toppled. Yellow sites are active, toppling sites. (Figure courtesy of M. Creutz.)

number, s, is the size of the avalanche. That is all! Fig. 3 shows snapshots of a propagating large avalanche.

The sandpile model has been generalized to represent earthquakes (24) or even starquakes (26). A modified version has been applied to economics (27). It was demonstrated that small shocks may lead to large avalanches of economic activity. The SOC behavior of these models has been documented by computer simulations. A few interesting analytical results have also been obtained, mostly by Deepak Dhar (28, 29). For example, Dhar was able to count the number of configurations in the stationary state. However, no analytical results for critical behavior, such as the value of the exponent τ, exist. Nevertheless, the sandpile picture has served well in providing an intuitive picture of SOC, traveling far beyond the physics community, and applied to many different kinds of situations. As Vice President Albert Gore has observed in his recent book (30), "The sandpile theory—self-organized criticality—is irresistible as a metaphor . . ."

III. Punctuated Equilibrium and SOC in Evolution

Some empirical observations indicate that biology operates in a critical state. This has been documented recently in great detail, and we refer the interested reader to that paper (31) for a more complete discussion. Fig. 4*A* shows the pattern of extinction events from the fossil history, as recorded by J. J. Sepkoski (32). There are long periods of relatively little activity interrupted by narrow intervals, or bursts, with large activity. Raup (15, 16, 33, 34) has plotted the data for the bursts as a histogram, as shown in Fig. 4*B*; this histogram suggests that the frequency of large events follows smoothly from the probability of small events. The histogram indicates that both large and small extinction events have a common dynamic origin. The distribution of extinction events of size s can roughly be interpreted as a power law, $P(s) \sim s^{-\tau}$, where $1 \leq \tau \leq 2$. Also, the distribution of lifetimes of fossil genera appears to obey a power law. The apparent punctuated equilibrium behavior both with respect to behavior of many species (or mass extinction events) and with respect to the behavior of single species indicates that both are collective effects involving many interacting degrees of freedom. They are two sides of the same coin.

A general theory or model of biological evolution must necessarily be abstract. It must, in principle, be able to describe all possible scenarios for evolution. For instance, it should be able to describe life on Mars, if it were to occur. This, of course, is an extremely precarious step. Only intuition can tell us what is important and what is not. The model cannot have any specific reference to actual species. It may, perhaps, not even refer to basic chemical processes or to DNA. It is precisely because of contingency that we cannot expect the theory to produce anything specific that is actually observed, in marked contrast to traditional theories. Biologists will complain that our models do not produce elephants.

We shall discuss a particularly simple toy model of an ecology of evolving species, the Bak–Sneppen (BS) model (18). The underlying picture is one where species interact with each other. When evolving to improve their fitness, presumably through random mutations, followed by selection of the fitter variants, they affect the fitness of other species in the global ecology. For a discussion of the general philosophy, see Kauffman's book (35).

For simplicity and bookkeeping, species are placed on a d-dimensional square lattice. The function of the lattice is to define who is interacting with whom: each species interacts with its $2d$ nearest neighbors. The one-dimensional case can be thought of as a food chain. Initially, the species are assigned random numbers, f_i, from 0 to 1. f_i represents the fitness of species i. It does not really matter what the starting point is. At each step the site i with the lowest fitness is chosen, and its

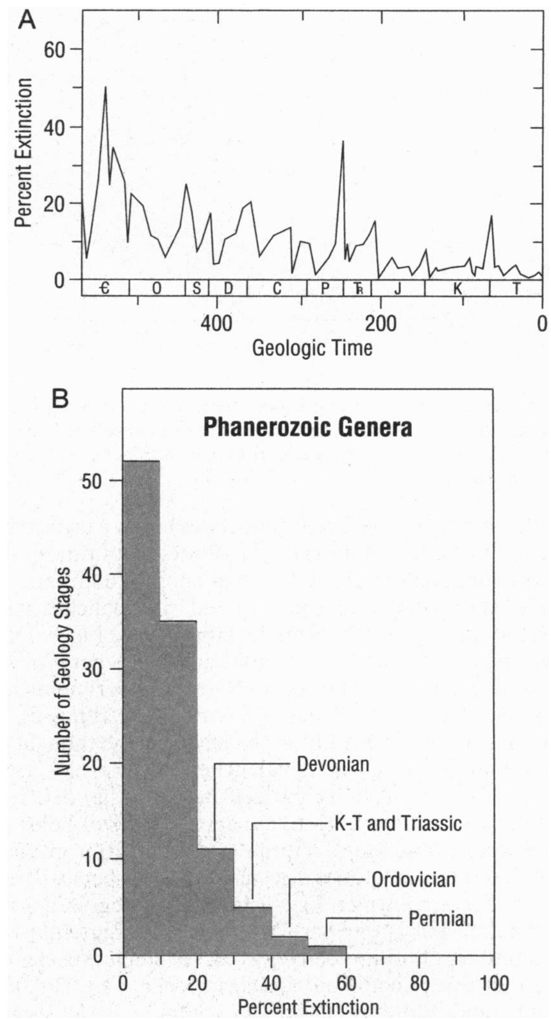

FIG. 4. (*A*) Extinction events recorded over 600 million years [reproduced from Sepkoski (32) with permission (copyright *Paleobiology*)]. The curve shows the estimated percentage of species that became extinct during consecutive intervals of 5 million years. Geologic periods are indicated: Є, Cambrian; O, Ordovician, S, Silurian; D, Devonian; C, Carboniferous; P, Permian; TR, Triassic; J, Jurassic; K, Cretaceous; T, Tertiary. (*B*) Histogram of the same events [reproduced from Raup (15) with permission (copyright AAAS)].

number f_i is replaced by a different random number, which is somewhere between 0 and 1. This step represents either a mutation to a different species, or the extinction of a species followed by replacement of another species in the same ecological niche. It mimics the Darwinian principle (36) that the least fit species become extinct. The random numbers at the $2d$ nearest neighbor sites are also replaced with new random numbers between 0 and 1. The fitness of the neighbors is contingent upon the properties of the species with which they interact. Thus, their happy and stable life (with maybe high random numbers) might become undermined by weak neighbors, so that they become next in line for extinction.

The model is so general that it can also be thought of as a model for macroeconomics. The individual sites represent economic agents, and the random numbers f_i represent their "utility functions." Agents modify their behavior to increase their wealth. The agents with lowest utility functions disappear and are replaced by others. This, in turn, affects other agents and changes their utility functions.

What could be simpler and have less structure than replacing some random numbers with other random numbers? Despite the simplicity of the model, its analysis is amazingly rich. After many updates have occurred, the ecology reaches a state in

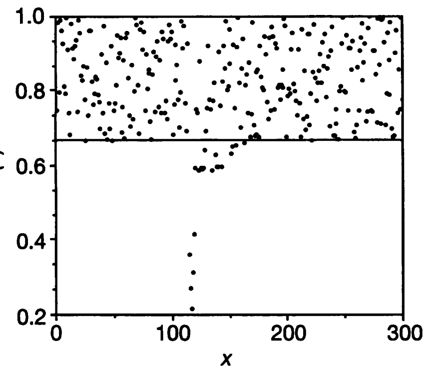

FIG. 5. Snapshot of f vs. position x during an avalanche in the evolution model. Most f values are above the critical value. The cluster of active sites with $f < f_c$ participate in the avalanche and undergo frequent changes.

which the density of species with fitnesses below a critical value f_c is zero. Species are uniformly distributed with fitness above f_c. No random number above f_c is ever chosen to mutate on its own. This stationary state is punctuated by avalanches, where, locally, the random numbers are less than f_c; see Fig. 5. During an avalanche, a great deal of rapid activity occurs in which species come and go at a fast pace. Nature "experiments" until it finds another "stable" ecology with high fitnesses. The Cambrian explosion 500 million years ago can be thought of as the grandmother of all such avalanches.

Fig. 6 shows the activity pattern in the 1d model. It is a fractal in space and time. Note the appearances of holes of all sizes between subsequent returns of activity to a given site. These holes represent periods of stasis where a species does not mutate or become extinct. Fig. 7 shows the integrated activity along the time axis at one particular site. This figure illustrates punctuated equilibrium behavior for a single species. We believe that this punctuated equilibrium behavior, first noted by Gould and Eldredge (7, 8), is common to all complex dynamical systems. The punctuations for single species are correlated to the avalanches in the global ecology.

In contrast to real biological evolution, in the computer evolution model the tape of Life may be rerun many times. It is easy to go back and find the event which triggered the punctuation starting at $s \approx 100$ in Fig. 7. What would have

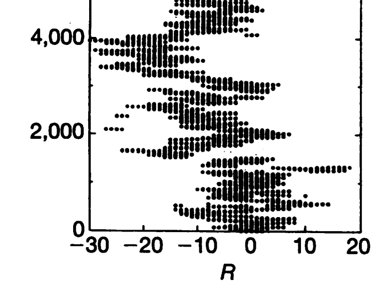

FIG. 6. Fractal cluster of activity in the 1d evolution model. The horizontal axis is a row of lattice sites; the vertical axis is time, s, or the number of update steps (22, 23).

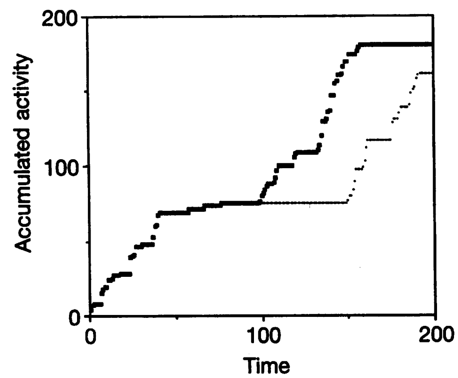

FIG. 7. Accumulated number of changes, or mutations, at a single site in the stationary state. The curve exhibits punctuated equilibrium behavior, with periods of stasis interrupted by intermittent bursts. The large bursts are correlated with large avalanches. The lower curve shows the result of rerunning the tape of evolution. A single update event, at the beginning of a large avalanche, was cancelled. This led to dramatically different behavior in the replay, demonstrating contingency.

happened if that event had not occurred? Thus, we simply skip one update step in our simulation but choose the same random numbers as before for all subsequent updates. Fig. 7 also shows the alternative run. Indeed, this change prevented the large avalanche, but other disasters happened instead, at a different point in time. In a noncritical, noninteractive biology, the effects of contingency are much less dramatic. So if we study the evolution model during the transient period before the system reaches criticality, we find evolution to be gradual, without large punctuations, and rerunning the tape with small modifications leads to the same history. In a noncritical biology, a meteor may not have been sufficient to trigger a major extinction event.

IV. Analytical Results

The evolution model is fascinating to theorists wanting to understand self-organized criticality. In contrast to sandpile and earthquake models, it has yielded to analytical mathematical efforts. These efforts include mean-field results by Flyvbjerg and coworkers (37, 38), a conjecture that the model can be related to Reggeon field theory (39) from high-energy physics that was made independently by us (19) and by Ray and Jan (25), and, finally, exact results, which we discuss here. In collaboration with Sergei Maslov, we have derived three fundamental equations that describe the process of self-organization (19), the hierarchical structure of the avalanches, and the stationarity condition (20, 21), respectively.

The Self-Organization Process: The Gap Equation. The critical stationary state is approached algebraically, through transient states. Let us consider the situation where the distribution of f values initially is uniform in the interval 0 to 1 in a d-dimensional system of linear size L. The first value of f to be chosen for updating is $\mathcal{O}(L^{-d})$. Eventually, after s time steps, a gap $G(s)$ opens up in the distribution of f values. We define the current gap, $G(s)$, to be the maximum of all minimum random numbers chosen, $f_{\min}(s')$, for all $0 \leq s' \leq s$. Fig. 8 shows $f_{\min}$ as a function of s during the transient for a small system. The full line shows the gap $G(s)$ as a monotonically increasing function of s. By definition, the separate instances when the gap $G(s)$ jumps to its next higher value are separated by avalanches. The average size of the jump in the gap at the completion of each avalanche is $[1 - G(s)]/L^d$. Consequently, the growth of the gap versus time, s, obeys the following gap equation (19):

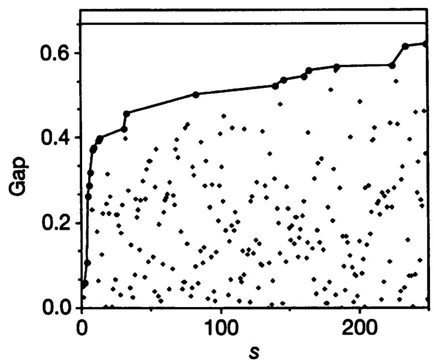

FIG. 8. The self-organization process in a small system. f_{min} vs. time s is shown (crosses). The full curve shows the gap, max[$f_{min}(s)$]. The gap approaches the critical value f_c asymptotically. Subsequent points (●) where the gap increases are separated by avalanches during which f_{min} is lower. On average, the avalanche size grows as the critical value is approached.

$$\frac{\partial G(s)}{\partial s} = \frac{1-G(s)}{L^d \langle S \rangle_{G(s)}}. \quad [1]$$

As the gap increases, so does the average avalanche size $\langle S \rangle$, which eventually diverges as $G(s) \to f_c$, whereupon the model is critical and the process achieves stationarity. In the limit $L \to \infty$, the density of sites with $f < f_c$ vanishes, and the distribution of f values is uniform above f_c. The gap equation **1** defines the mechanism of approach to the self-organized critical attractor. It contains the essential physics of SOC phenomena. When the average avalanche size diverges, $\langle S \rangle \to \infty$, the system becomes critical. At the same time, $\partial G/\partial s$ approaches zero, which means that the system becomes stationary. When interpreting actual biology, we assume that the transient took place long ago, so that throughout the evolutionary history that is observed in the fossil record, biology has been critical.

In order to solve the gap equation we need to determine precisely how the average avalanche size $\langle S \rangle_{G(s)}$ diverges as the critical state is approached. Numerically, we find $\langle S \rangle_{G(s)} \sim (f_c - f_o)^{-\gamma}$, with $\gamma \approx 2.7$ in one dimension. Inserting this into Eq. **1** and integrating, we find

$$\Delta f = f_c - G(s) \sim (s/L^d)^{-\frac{1}{\gamma-1}}, \quad [2]$$

which shows that the critical point ($\Delta f = 0$) is approached algebraically with an exponent 0.58 in one dimension.

Hierarchy of Avalanches: The Gamma Equation. Consider the stationary SOC state, and let $\mathcal{P}(f)$ be the probability to have an f avalanche separating consecutive points in time where the minimum random number chosen is greater than f. An avalanche of spatial extent r, by definition, leaves on average $\langle r^d \rangle$ sites with new uncorrelated random numbers between f and 1. If f is increased by a small amount df, the differential in the probability that an $f + df$ avalanche will not end at the same time as the f avalanche is determined by the probability that any of the new random numbers generated by the avalanche fall within df of f. This probability is $df\langle r^d \rangle/(1 - f)$. We thus obtain the rigorous result, the "gamma" equation:

$$\frac{d(\ln \mathcal{P})}{df} = \frac{\langle r^d \rangle_f}{1-f}, \text{ for } f < f_c. \quad [3]$$

Since close to f_c, $\mathcal{P}(f) \sim \Delta f^\gamma$, where $\Delta f = f_c - f$, the fundamental relation (20, 21)

$$\gamma = \frac{\langle r^d \rangle \Delta f}{1-f} \quad [4]$$

holds for $\Delta f \to 0$. Surprisingly, the quantity γ which enters in the gap equation, here appears as a constant rather than a critical exponent. It is the number of random numbers between f and f_c left behind by an f avalanche that has died. Thus $\langle r^d \rangle \sim \Delta f^{-\gamma_\perp}$, where $\gamma_\perp = 1$. The gamma equation (Eq. **4**) gives a convenient way to determine the critical point accurately. When $(1 - f)/\langle r^d \rangle$ is plotted vs. f, the slope is equal to $1/\gamma$ and the intersection with the f axis is f_c. We find $\gamma \approx 2.7$ and $f_c = 0.66695 \pm 0.00005$ (Fig. 9). Thus, contrary to early speculations, it is unlikely that the critical f_c is exactly 2/3.

Stationarity: $\eta = 0$. The next equation utilizes the stationarity of the process to relate the activity within running avalanches to the size distribution of avalanches (20, 21). Suppose that at some point in time an avalanche starts by "injecting" a single site into the gap; i.e., initially only one site has fitness less that f_c. On average, how large is the number $\langle n(s) \rangle$ of active sites after s update steps? Active sites are defined to be those sites that have random numbers less than f_c, and the average is taken over all avalanches, including those that die out.

This question has an interesting similarity with the following, seemingly unrelated, problem: suppose a gambler in Las Vegas plays on a "fair" roulette wheel, where the probabilities of "red" and "black" are each 1/2. He starts with one dollar, and as long as he has money left, he plays one dollar at each run of the roulette. How much money can he expect to have left after s roulette runs? In both cases, the answer is essentially one unit. Both processes are stationary critical branching processes. At each point in time, stationarity demands that the average number of active sites is constant, so that (20, 21)

$$\langle n(s) \rangle \sim s^\eta, \text{ with } \eta = 0. \quad [5]$$

This utterly simple but very deep equation is the "eta" equation. In spite of its simplicity, it is a highly nontrivial result; critical branching processes, in general, do not have $\eta = 0$. Each injected particle gives rise to an avalanche, with size distribution $s^{-\tau}$. The average activity, over all avalanches, after s steps is the product of the activity of surviving avalanches and the probability for an avalanche to survive s steps. One consequence of the eta equation is that the growth of activity in surviving avalanches, $n(s)$, must exactly compensate the avalanches that have died; i.e., $n(s) \sim s^{-\tau+1}$. Thus, the internal structure of the avalanches is related to the duration of avalanches. For the gambler, the longer he plays, the less is his chance of survival. This is compensated by the fact that if he survives, his return is growing bigger. The exponent τ for the gambler is 3/2. For the $1d$ evolution model the exponent is 1.1; in two dimensions it is 1.26.

We have conjectured (19) that the exponent τ for the evolution model is the same as the exponent τ in Reggeon field theory, a quantum field theory from high-energy physics. Fig.

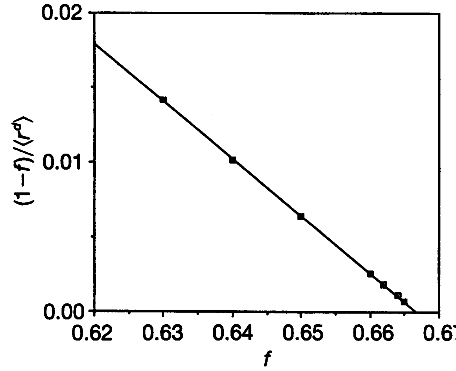

FIG. 9. Plot of $(1 - f)/\langle r^d \rangle$ vs. f in the $1d$ evolution model. The inverse slope gives the exponent f axis gives $f_c = 0.66695$.

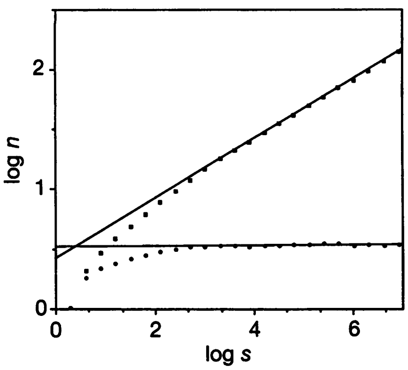

FIG. 10. Average number of active sites $\langle n \rangle$ s time steps after the initiation of a single avalanche (lower curve) in the $2d$ evolution model. The curve approaches a constant, indicating $\eta = 0$. The upper curve shows the (larger) number of active sites in surviving avalanches, $n(s) \sim s^{-\tau+1}$ (20, 21).

10 also shows the growth of activity in surviving clusters, increasing with the exponent $\tau = 1.26$. Actually, for the random neighbor model studies by Flyvbjerg *et al*, (37) and by de Boer *et al.* (38), the exponent is also 3/2. The equivalence of the situation of the gambler with evolution, in "mean-field," is not accidental. We survive by chance and coincidence rather than by merit. Whatever fairness exists is statistical; both the fair roulette wheel and evolution have $\eta = 0$.

These equations, and other general considerations, lead to an alphabet soup of scaling relations for various physical quantities, such as the fractal dimension of the avalanche, D, in Fig. 6, the fractal dimension of active sites which have fitnesses less than the critical value, d_f, and the power spectrum of the local activity, $S(f)$. All of the critical exponents can be expressed in terms of two fundamental exponents, D and τ. These two independent exponents are possibly given by Reggeon field theory. In conclusion, we have a rather complete description of the self-organization process and the resulting critical properties in the stationary attractor. The hope is that insight derived from the detailed study of this particular model can be extended to produce a more general phenomenological theory of complexity in nature. That theory will not be beautiful; it trivializes all the nuances and details that make complex systems exciting for humans. These details become just one possible realization among many other possibilities allowed by the theory.

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