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# INTUITIONISM: IS SOMETHING WRONG WITH OUR LOGIC?

> The long belief in the universal validity of the principle of excluded third in mathematics is considered by intuitionism as a phenomenon of history of civilization of the same kind as the old-time belief in the rationality of $\pi$ or in the rotation of the firmament on an axis passing through the earth. And intuitionism tries to explain the long persistence of this dogma by . . . the practical validity . . . of classical logic for an extensive group of *simple everyday phenomena*. [This] fact apparently made such a strong impression that . . . classical logic . . . became a deep-rooted habit of thought which was considered not only as useful but as a priori.
>
> I hope I have made clear that intuitionism on the one hand subtilizes logic, on the other hand denounces logic as a source of truth. Further that intuitionistic mathematics is inner architecture, and that research in the foundations of mathematics is inner inquiry . . .
>
> (Brouwer 1948: 94, 96)

## 1. Revising Classical Logic

THE practice of mathematics is primarily a *mental* activity. To be sure, mathematicians use paper, pencils, and computers, but at least in theory these are dispensable. The mathematician's main tool is her mind. Although the philosophies considered in this chapter are quite different from (and even incompatible with) each other, they all place emphasis on this activity of mathematics, paying attention to its basis or justification. A central theme uniting the views is a rejection of certain modes of inference in mathematics (see also ch. 1, §2). The philosophies considered here demand revisions to the mathematics of their day, and our day.

The main item is the *law of excluded middle* (LEM), sometimes called the 'law of excluded third' and 'tertium non datur' (TND). Let $\Phi$ be a proposition. Then the corresponding instance of excluded middle is the proposition that either $\Phi$ or it is not the case that $\Phi$, sometimes abbreviated as $\Phi$ or not-$\Phi$, or in symbols $\Phi \vee \neg\Phi$. In semantics, the closely related *principle of bivalence* is that every proposition is either true or false, and so there are only two possible truth-values—hence the name 'excluded middle'.[1] *Intuitionism* is a general term for philosophies of mathematics that demur from excluded middle.

Common logical systems that include excluded middle are called *classical*, and mathematics pursued with classical logic is called *classical mathematics*. The weaker logic, without excluded middle, is called *intuitionistic logic*, and the corresponding mathematics is *intuitionistic mathematics*. See Dummett 1977 for details.

Intuitionistic logic lacks other principles and inferences that rely on excluded middle. One of these is the law of double negation elimination, which allows one to infer a proposition $\Phi$ from the denial of the denial of $\Phi$. Using intuitionistic logic, one can infer not-not-$\Phi$ from $\Phi$, but not conversely. Suppose that someone derives a contradiction from a proposition in the form not-$\Phi$. Then both the classical mathematician and the intuitionist will conclude that not-not-$\Phi$ (via *reductio ad absurdum*). The classical logician will also infer (the truth of) $\Phi$, but this last inference is disallowed in intuitionistic logic (unless the mathematician already knows that $\Phi$ is either true or false).

To take another example, suppose that a mathematician proves that not all natural numbers have a certain property $P$. In symbols, the theorem is $\neg\forall xPx$. A classical mathematician would then infer

[1] Excluded middle and bivalence are equivalent if one assumes the platitudes that for any proposition $\Phi$, $\Phi$ is true if and only if $\Phi$, and $\Phi$ is false if and only if $\Phi$ is not true. These principles are sometimes called 'Tarski biconditionals' or 'T-sentences'.

that *there is* a natural number that lacks $P$ (i.e. $\exists x \neg Px$). The intuitionist would not allow this conclusion (in general). Readers familiar with mathematical logic are invited to check that an inference from $\neg \forall x Px$ to $\exists x \neg Px$ relies on excluded middle or some equivalent principle or inference.

The proposed, or demanded, revisions to logic are tied to philosophy. Intuitionists argue that excluded middle and the related inferences indicate a belief in the independent existence of mathematical objects and/or a belief that mathematical propositions are true or false independent of the mathematician. In present terms, intuitionists argue that excluded middle is a consequence of realism in ontology and/or realism in truth value (see ch. 2, §§2.1, 2.2). Some intuitionists reject this realism outright, while others just argue that mathematics should not presuppose any such metaphysical thesis.

The mathematics one gets via intuitionistic restrictions is very different from classical mathematics (see, for example, Heyting 1956, Bishop 1967, Dummett 1977). Critics commonly complain that the intuitionistic restrictions cripple the mathematician. On the other hand, intuitionistic mathematics allows for many potentially important distinctions not available in classical mathematics, and is often more subtle in interesting ways. Here we examine what leads some philosophers to demand the restriction.

## 2. The Teacher, Brouwer

Although Hilbert's finitary arithmetic had a clear and explicit Kantian influence (see ch. 6, §3), the previous two chapters have recorded a marked trend away from Immanuel Kant's philosophy of mathematics. Of all the twentieth century authors considered in this book, L. E. J. Brouwer was the most Kantian. Brouwer (1912: 78) dubs Kant's philosophy 'an old form of intuitionism' (although Kant was not critical of the practice of mathematics). It is thus no coincidence that Hilbert's finitary arithmetic has an affinity with intuitionistic mathematics. Brouwer and Hilbert both noted that if one sticks to the practice of finitary arithmetic, there is not much difference between the classical and intuitionistic approach. There are, however, substantial and irreconcilable differences between

Hilbert and Brouwer. They clearly disagree over what Hilbert calls ideal mathematics, which, of course, is the bulk of mathematics. More important here, the philosophical background to their enterprises could hardly be more different.

In a paper comparing intuitionism with formalism, Brouwer (1912: 77) noted that scientific principles 'can only be understood to hold in nature with a certain degree of approximation', and he pointed out that the main 'exceptions to this rule have from ancient times been practical arithmetic and geometry . . .'. Mathematics has 'so far resisted all improvements in the tools of observation'. The philosophical problem is to explain the exactitude enjoyed by mathematics, and its resistance to empirical refinement. Intuitionists and formalists differ on the source of the 'exact validity' of the mathematical sciences: 'The question where mathematical exactness does exist, is answered differently by the two sides; the intuitionist says: in the human intellect; the formalist says: on paper.'

For Brouwer, as for Kant, most mathematical truths are not capable of 'analytic demonstration'. They cannot become known by mere analysis of concepts, and they are not true in virtue of meaning. So the bulk of mathematics is *synthetic*. Yet mathematical truth is a priori, independent of any particular observations or other experience we may have. Brouwer held that mathematics is mind-dependent, concerning a specific aspect of human thought. In the terminology of chapter 2, §2, Brouwer was an anti-realist in ontology and an anti-realist in truth-value. And he was no empiricist. Like Kant, Brouwer tried to forge a synthesis between realism and empiricism.

For Kant and for Brouwer, 'the possibility of disproving' mathematical laws experimentally is 'not only excluded by a firm belief, but [is] entirely unthinkable'. For Brouwer, mathematics concerns the ways humans approach the world. To think at all is to think in mathematical terms.[2]

Brouwer (1912: 77) echoes the major Kantian theme that a human being is not a passive observer of nature, but rather plays an *active* role in organizing experience: 'that man always and everywhere creates order in nature is due to the fact that he not only

[2] As we saw in §3 of the previous chapter, Hilbert said something similar about mathematics, but Hilbert's statement was limited to the use of *symbols* in reasoning. As we will see, for Brouwer, the symbols are a side-matter, well removed from the essence of mathematics.

isolates the causal sequences of phenomena . . . but also supplements them with phenomena caused by his own activity . . .'. Mathematics concerns this active role.

Brouwer conceded that developments in nineteenth-century mathematics made the Kantian view of *geometry* untenable. The advent of rigour, leading to the idea of logical consequence as independent of content, and the development of the multiple interpretations of projective geometry, supported the thesis that only the logical form of a geometric theorem matters (see ch. 6, §2). This left no room for 'pure intuition' in geometry. According to Brouwer, the main blow to the Kantian idea that geometry concerns synthetic a priori forms of perception was the advent and acceptance of non-Euclidean geometry: 'this showed that the phenomena usually described in the language of elementary geometry may be described with equal exactness . . . in the language of non-Euclidean geometry; hence, it is not only impossible to hold that the space of our experience has the properties of Euclidean geometry but it has no significance to ask for *the* geometry which would be true for the space of our experience' (Brouwer 1912: 80). This point was also made by Henri Poincaré (1903: 104), another mathematician with intuitionistic leanings (see Shapiro 1997: ch. 5, §3.1).

Thus, Brouwer abandoned Kant's view of *space*. In its place, he made a courageous proposal to found all of mathematics on a Kantian view of *time*. Difficult passages like the following occur throughout Brouwer's writing:

> [Modern intuitionism] considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomena of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely. (Brouwer 1912: 80)

This seems to defy sharp interpretation. The underlying idea might be to base the natural numbers on the forms of temporal perception, just as Kant founded geometry on the forms of spatial perception. We apprehend the world as a series of distinct moments. Each moment gives rise to another one. This is the 'bare two-oneness'. And the second moment gives way to a third, and so on, thus yielding the natural numbers.

Brouwer states that this 'basal intuition' unites the 'connected and separate'. Each moment is unique, and yet is connected to every other moment. The original intuition also unites the 'continuous and the discrete' and 'gives rise immediately to the intuition of the linear continuum'. The moments of time are distinct, and yet they flow continuously. Brouwer mentions that the notion of 'between' leads to the rational and, ultimately, real numbers. The idea seems to be that we know *a priori* that between any two moments, there is a third. The temporal continuum 'is not exhaustible by the interposition of new units and . . . therefore [cannot] be thought of as a mere collection of units'. So both the natural and real numbers—the discrete and the continuous—are grounded in temporal intuition. This yields arithmetic and real analysis.

Brouwer then follows standard Cartesian techniques to found geometry on the real numbers, by identifying a point with a pair of numbers. Brouwer claims that this qualifies ordinary plane and solid geometry, as well as non-Euclidean and *n*-dimensional geometry, as synthetic a priori.[3] Even geometry is ultimately based on the intuition of time.

Recall that for Kant, arithmetic and geometry are not analytic because they rely on 'intuition'. As noted in chapter 4, §2, there is substantial disagreement among scholars concerning exactly what Kantian intuition is. In the treatment there, I suggested that a central component of Kant's a priori mathematical intuition is *construction*. In particular, the crucial intuitive (and synthetic) aspects of a Euclidean demonstration are the 'setting out', where a typical figure satisfying the hypothesis is drawn, and the auxiliary constructions, where the reader is instructed to draw additional lines and/or circles on the given figure. Clearly, these constructions are not physical operations on paper or a blackboard, but are idealizations thereof. One cannot literally draw a line with no thickness. For Kant, Euclid's 'construction' is a mental

[3] Recall that Frege held that arithmetic and analysis are analytic, and he maintained a Kantian view of geometry as the synthetic a priori forms of space. Thus, Frege would not accept the Cartesian foundation of geometry on arithmetic and analysis. He was thus the exact opposite of Brouwer.

act, the mind's active process of apprehending the forms of perception.

Brouwer is quite explicit that the essence of mathematics is idealized mental construction. Consider, for example, the proposition that for every natural number $n$, there is a prime number $m > n$ such that $m < n! + 2$ and $m$ is prime. For Brouwer, this proposition invokes a *procedure* that, given any natural number $n$, produces a prime number $m$ that is greater than $n$ but less than $n! + 2$. The mathematician has not established this proposition until she has given such a procedure. Brouwer (1912, 87–8) discusses a version of the Schröder–Bernstein theorem: if there is a one-to-one correspondence between set $A$ and a set divided into three disjoint parts $A_1 + B_1 + C_1$ such that there is a one-to-one correspondence between $A$ and $A_1$, then there is also a one-to-one correspondence between $A$ and $A_1 + B_1$. This theorem is provable in classical mathematics, indeed in second-order logic (see Shapiro 1991: 102–3). However, Brouwer wrote that the intuitionist interprets the proposition as follows:

> if it is possible, *first* to construct a law determining a one-to-one correspondence between the mathematical entities of type $A$ and those of type $A_1$, and *second* to construct a law determining a one-to-one correspondence between the mathematical entities of type $A$ and those of $A_1$, $B_1$, and $C_1$, then it is possible to determine from these two laws by means of a finite number of operations a third law, determining a one-to-one correspondence between the mathematical entities of type $A$ and those of types $A_1$ and $B_1$.

The classical theorem concerning the existence of the one-to-one correspondence does not yield the requisite *procedure*. Brouwer argued that it is unlikely that the Schröder–Bernstein theorem is provable, since we do not know a general method of producing the procedure of the conclusion.

Brouwer's repudiation of excluded middle flows from his constructive conception of mathematics. Consider first the inference of double negation elimination, the classical rule that allows one to infer a sentence $\Phi$ from a premiss that it is not the case that it is not the case that $\Phi$. Let $P$ be a property of natural numbers and consider a proposition that there is a number $n$ such that $P$ holds of $n$; in symbols this is $\exists nPn$. For an intuitionist, this proposition is established only when one shows *how to construct* a number $n$ that has the property $P$. The negation of a proposition $\Phi$, symbolized $\neg\Phi$ is established when one shows that the assumption of (the construction corresponding to) $\Phi$ is contradictory. Thus, the double negation $\neg\neg\exists nPn$ is established when one shows that an assumption $\neg\exists nPn$ is contradictory. Clearly, to derive a contradiction from the assumption that $\neg\exists nPn$ is not to construct a number $n$ such that $Pn$. Indeed, we can derive the contradiction and have no idea what such a number $n$ might be. Thus, from Brouwer's perspective, double negation elimination is invalid.

The corresponding instance of excluded middle is that either there is or is not a number $n$ such that $Pn$. To establish this instance, one would have either to construct a number $n$ and then show $Pn$ or else derive a contradiction from the assumption that $\exists nPn$. Throughout his career, Brouwer tirelessly argued that we have no a priori reason to believe this principle holds in general.

Brouwer (1948: 90) concedes that classical (real and complex) analysis may be 'appropriate . . . for science', but he argues that it has 'less mathematical truth' than intuitionistic analysis, since classical analysis runs against the mind-dependent nature of mathematical construction. This is a bold divorce between mathematics and the empirical sciences.

Brouwer traces the belief in excluded middle to an incorrect and outdated philosophy of mathematics, the view that I call 'realism in ontology'. He argues that the 'various ways' in which classical mathematics is justified 'all follow the same leading idea, viz., the presupposition of the existence of a world of mathematical objects, a world independent of the thinking individual, obeying the laws of classical logic . . .' (Brouwer 1912: 81). Someone who holds that the natural numbers, say, exist independent of the mathematician is likely to interpret the foregoing instance of excluded middle as 'either the collection of natural numbers contains a number $n$ such that $Pn$ or it does not'. From that perspective, every instance of excluded middle is obvious, indeed a logical truth.

Recall that Plato was critical of the geometers for using dynamic language, speaking of 'squaring and applying and adding and the like . . .'. He insisted that 'the real object of the entire subject is . . . knowledge . . . of what eternally exists, not of anything that comes to be this or that at some time and ceases to be' (*Republic*, Book 7, see ch. 1, §2, and ch. 3, §2 above). Clearly, Brouwer would side with the geometers against Plato. Mathematics concerns mental activity, not some ideal realm of independently existing entities. As such, the language should be dynamic, not static.

On Brouwer's view, the practice of mathematics flows from *introspection* of one's mind. In philosophy, a slogan of traditional idealism is: 'to exist is to be perceived.' A corresponding slogan for intuitionism would be that in mathematics, 'to exist is to be constructed'. It follows from Brouwer's view that all mathematical truths are accessible to the mathematician, at least in principle: 'The . . . point of view that there are no non-experienced truths . . . has found acceptance with regard to mathematics much later than with regard to practical life and to science. Mathematics rigorously treated from this point of view, including deducing theorems exclusively by means of introspective construction, is called intuitionistic mathematics' (Brouwer 1948: 90). According to Brouwer, the classical mathematician incorrectly 'believes in the existence of unknowable truths'.

For Brouwer, every legitimate mathematical proposition directly invokes human mental abilities. Mathematical assertions are 'realized, i.e. . . . convey truths, if these truths have been experienced'. Thus, as understood by an intuitionist, the principle of excluded middle amounts to a principle of *omniscience*: 'Every assignment . . . of a property to a mathematical entity can be judged, i.e., proved or reduced to absurdity.' Brouwer's argument is that we are not omniscient and so we should not assume excluded middle.

Recall that a definition of a mathematical entity is *impredicative* if it refers to a collection that contains the entity (ch. 1, §2, and ch. 5, §2). For example, the usual definition of 'least upper bound' is impredicative, since it characterizes a number in terms of a collection of upper bounds, and the defined number is a member of that collection. For a realist in ontology, impredicative definitions are innocuous, since there is no problem in characterizing an objectively existing entity in terms of a collection that contains the entity. For a realist, there is no more problem with 'least upper bound' than with the similarly impredicative 'most stubborn member of the faculty'. For an intuitionist, however, an impredicative definition is viciously circular. We cannot *construct* a mathematical entity by using a collection that contains the entity.

In similar fashion, Brouwer (1912: 82) objects to consideration of collections of mathematical entities, as if they were completed totalities. He complains that a classical mathematician . . . 'introduces various concepts entirely meaningless to the intuitionist, such as for instance "the set whose elements are the points of space", "the set whose elements are the continuous functions of a variable", "the set whose elements are the discontinuous functions of a variable", and so forth'. For the intuitionist, we are never finished constructing all of the elements of one of these collections, and so we cannot speak of 'the set' of such elements.

Brouwer's conception of the nature of mathematics and its objects leads to theorems that are (demonstrably) false in classical mathematics. As classically conceived, a real number can be thought of as an infinite decimal, a completed infinity. As Brouwer (1948) put it, the classical mathematician holds that 'from the beginning the $n^{th}$ element is fixed for each $n$'. Moreover, any arbitrary or random sequence of digits is a legitimate real number. Early in his career, Brouwer identified real numbers with decimal expansions given *by a rule*: 'Let us consider the concept: "real number between 0 and 1" . . . For the intuitionist [this concept] means "law for the construction of an elementary series of digits after the decimal point, built up by means of a finite series of operations"' (Brouwer 1912: 85). For technical reasons, a focus on decimal expansions proved to be awkward and, in any case, it is more common for mathematicians to speak of Cauchy sequences of rational numbers, rather than decimal expansions. In these terms, for the early Brouwer, only Cauchy sequences given by rules determine legitimate real numbers.[4]

Later, however, Brouwer supplemented these rule-governed sequences with what are sometimes called 'free choice sequences'. Brouwer envisioned a 'creative subject' with the power to freely produce further members of an evolving choice sequence (or, ignoring the technicality, further digits of a decimal expansion). Free choice sequences do not have the aforementioned property, ascribed to classical real numbers, 'from the beginning the $n^{th}$

[4] A sequence $a_1, a_2, \ldots$ of rational numbers is *Cauchy* if for each rational number $\varepsilon > 0$ there is a natural number $N$ such that for all natural numbers $m$, $n$, if $m > N$ and $n > N$ then $-\varepsilon < a_m - a_n < \varepsilon$. A Cauchy sequence is given by a rule only if there is an effective procedure for calculating the members $a_n$, and an effective procedure for calculating the bound $N$, given $\varepsilon$. The principle of completeness is that every Cauchy sequence converges to a real number.

element is fixed for each $n$'. The key feature of both rule-governed and free choice sequences is that each one is only a potential infinity, not an actual infinity. We never have the entire sequence before us, as it were. We only have the ability to continue the sequence as far as desired, either by following the rule or by having the creative subject continue to elaborate a free choice sequence.

From this perspective, any theorems about a given real number must follow from a finite amount of information about it. For a rule-governed sequence, the mathematician can use the rule to establish facts about the corresponding real number. For a free choice sequence, however, there is no rule, and so the only information the mathematician ever has about it—at any point in time—consists of a finite initial segment of the sequence. Let $a$ be a free choice sequence. It follows that any property $P$ that a mathematician ascribes to $a$ must be based on a finite initial segment of a corresponding Cauchy sequence. That is, the mathematician should never have to determine the entire sequence for $a$ before she is able to determine whether $P$ holds of $a$—simply because the entire sequence never exists. Thus, if $a$ has a property $P$, then there is a rational number $\varepsilon > 0$ such that if a real number $b$ is within $\varepsilon$ of $a$, then $P$ holds of $b$ as well. Using similar reasoning, Brouwer established that every function from real numbers to real numbers is (uniformly) continuous![5]

The proof of this theorem makes essential use of free choice sequences. If only rule-governed real numbers are considered, then discontinuous functions cannot be ruled out on logical grounds. However, the existence of discontinuous functions entails unwanted instances of excluded middle. For example, let $f$ be any function such that for all real numbers $x$, $fx = 0$ if $x \leq 0$ and $fx = 1$ if $x > 0$. So $f$ has a discontinuity at 0. Now define a Cauchy sequence

[5] Incidentally, it follows from Brouwer's theorem that the axiom of choice fails in intuitionistic analysis. One formulation of this axiom is that if for every real number $a$ there is a real number $b$ such that a given relation R holds between $a$ and $b$, then there is a function $f$ such that for every $a$, the relation R holds between $a$ and $fa$. The function $f$ picks out (or 'chooses') a value $b$. In intuitionistic analysis, it is provable that for every real number $a$ there is a natural number $b$ such that $b > a$. We need only approximate $a$ to within, say, .5 and then pick a natural number much larger than that approximation. However, there cannot be a continuous function $f$ such that for every real number $a$, $fa$ is a natural number and $fa > a$.

$< a_n >$ as follows: if there is no counterexample to the Goldbach conjecture less than $n$, then $a_n = 1/n$; otherwise let $a_n = 1/p$, where $p$ is the smallest such counterexample. For an intuitionist, $< a_n >$ is a legitimate Cauchy sequence (since we can effectively calculate each member, and effectively determine arbitrarily close approximations—see note 4 above). Let $a$ be the real number that $< a_n >$ converges to. Notice that $a > 0$ if and only if the Goldbach conjecture is false. What of the real number $fa$? We have that $fa = 0$ if the Goldbach conjecture is false and $fa = 1$ otherwise. So one cannot approximate $fa$ to within .4 unless one knows whether the Goldbach conjecture is true. Thus, if $f$ were a legitimate function, then either the Goldbach conjecture is true, or it is not the case that the Goldbach conjecture is true. This last is an unwanted instance of excluded middle (at least until the Goldbach conjecture is settled, in which case we will use another example).

This argument is an instance of the so-called 'method of weak counterexamples', where the intuitionist demurs from a certain principle of classical mathematics (the existence of discontinuities in this case) by showing that the principle entails instances of excluded middle. To take another example, consider a (purported) function $g$ such that $gx = 0$ if $x$ is rational and $gx = 1$ if $x$ is irrational. Let $c$ be any real number. In order to determine whether $gc = 0$, one must determine whether $c$ is rational. If $c$ is a choice sequence, one *cannot* determine whether $c$ is rational. Recall that any information about a free choice sequence must be obtained from a finite segment of a corresponding Cauchy sequence. Any finite segment (or any finite decimal) can be continued to produce a rational and any finite segment can be continued to produce an irrational. If $c$ is rule-governed, then in some cases it may be possible to determine whether $c$ is rational and thus whether $gc = 0$, by reasoning about the rule. However, there is no general method for calculating $gc$. Again, the existence of $g$ entails unwanted instances of excluded middle. Thus, the definition of $g$ is not legitimate for an intuitionist.

In contrast to this, discontinuous functions are a staple of classical mathematics. They proved essential to physics (see, for example, Wilson 1993a) but, as noted above, Brouwer was not interested in tailoring mathematics to the needs of science.

Brouwer recognized that intuitionistic mathematics is not a mere restriction of classical mathematics, but is incompatible with it:[6] 'there are intuitionistic structures which cannot be fitted into any classical logical frame, and there are classical arguments not applying to any introspective image' (Brouwer 1948: 91). The reason concerns the basic differences in how the fields are construed:

> theorems holding in intuitionism, but not in classical mathematics, often originate from the circumstance that for mathematical entities . . . the possession of a certain property imposes a special character on their way of development from the basic intuition, and that from this special character of their way of development from the basic intuition, properties ensue which for classical mathematics are false.

In addition to, or along with, the trend away from Kant's philosophy of mathematics, the thinkers covered in the previous two chapters showed an increasing tendency to focus on the *language* and the *logic* of mathematics. Logicists set out to reduce mathematics to logic, claiming that mathematics is no more than logic, while formalists appealed to the practice of manipulating characters in rule-governed ways. Alberto Coffa (1991) calls this trend the 'semantic tradition', and Michael Dummett dubbed it the 'linguistic turn'. Brouwer bucked the trend. For him, language is no more than an imperfect medium for communicating mental constructions, and it is these constructions that constitute the essence of mathematics. Suppose that a mathematician accomplishes a mental construction and wants to share it with others. She writes some symbols down on paper and submits it to a journal. If all goes well with the editor and then with subsequent readers, other mathematicians can experience the mental, mathematical construction themselves, by reading the symbols in the journal. Like any other medium, however, language is fallible. The readers may not 'get it' in the sense that they may not experience any construction after reading the paper (or trying to), or they may experience a different construction from that of the first mathematician. In either case,

[6] There is a school of mathematics and philosophy, called 'constructivism', that accepts neither excluded middle nor the non-classical aspects of intuitionistic analysis. Roughly, Errett Bishop (1967) embraces only the common core of classical and intuitionistic mathematics. He insists on an epistemic understanding of the language of mathematics. To say that there exists a number with a given property, for example, one must give a method for finding such a number. Bishop calls excluded middle a principle of 'limited omniscience'.

the problem is not with the first mathematical construction. As in the film *Cool Hand Luke*, what we have here is (only) a failure to communicate. On Brouwer's view, logic is merely a codification of the rules for communicating mathematics via language.

Thus for Brouwer, logicism and formalism both focus on the external trappings of mathematical communication and completely ignore the essence of mathematics. He explicitly rejected the concern with consistency proofs:

> in . . . construction . . . neither the ordinary language nor any symbolic language can have any other rôle than that of serving as a non-mathematical auxiliary, to assist the mathematical memory or to enable different individuals to build up the same [construction]. For this reason the intuitionist can never feel assured of the exactness of a mathematical theory by such guarantees as the proof of its being non-contradictory, the possibility of defining its concepts by a finite number of words . . . or the practical certainty that it will never lead to a misunderstanding in human relations. (Brouwer 1912, 81)

In other words, the focus on language and logic misses the point.

## 3. The Student, Heyting

In some ways, Brouwer's student Arend Heyting was the more influential of the two—via a contribution that Brouwer did not approve, and even Heyting showed some ambivalence over. He developed a rigorous *formalization* of intuitionistic logic. The system is sometimes called *Heyting predicate calculus* (see, for example, Heyting 1956: ch. 7, or some contemporary textbooks in symbolic logic like Forbes 1994: ch. 10). Heyting 1930 suggested that from the underlying metaphysical assumptions—realism in truth-value—of classical logic, the language of classical mathematics is best understood in terms of (objective) *truth conditions*. A semantics for classical mathematics would thus delineate the conditions under which each sentence is true or false. With the rejection of bivalence (see §1 above), a semantics like this is inappropriate for intuitionism. Instead, intuitionistic language should be understood in terms of *proof conditions*. A semantics would delineate what counts as a canonical proof for each sentence. In rough terms, here are some clauses:

A proof of a sentence of the form 'Φ and Ψ' consists of a proof of Φ and a proof of Ψ.

A proof of a sentence of the form 'either Φ or Ψ' consists of either a proof of Φ or a proof of Ψ.

A proof of a sentence of the form 'if Φ then Ψ' consists of a method for transforming any proof of Φ into a proof of Ψ.

A proof of a sentence of the form 'not-Φ' consists of a procedure for transforming any proof of Φ into a proof of absurdity. In other words, a proof of not-Φ is a proof there can be no proof of Φ.

A proof of a sentence of the form 'for all $x$, $\Phi(x)$' consists of a procedure that, given any $n$, produces a proof of the corresponding sentence $\Phi(n)$.

A proof of a sentence of the form 'there is an $x$ such that $\Phi(x)$' consists of the construction of an item $n$ and a proof of the corresponding $\Phi(n)$.

The system is now known as *Heyting semantics* (see also Dummett 1977: ch. 1). Notice that one cannot have a canonical proof of a disjunction 'either Φ or Ψ' unless one has a proof of one of the clauses. So one cannot have such a proof of an instance of excluded middle 'Φ or not-Φ' unless one has either a proof of Φ or a proof that there can be no proof of Φ. So many instances of excluded middle do not seem to be justified by this semantics. Notice also that one cannot prove a sentence that begins 'there is an $x$' without showing how to produce such an $x$. This is a formalization of a major intuitionistic theme, shared by all schools of intuitionism.

It is ironic that Heyting's work here is anathema to Brouwer's attitude toward language and logic. Heyting's formal proposals might have been an attempt to be helpful to his classical colleagues, providing them with at least an outline of the linguistic trappings of intuitionistic mathematics. Heyting shared Brouwer's views concerning the prevalence of mental construction and the *downplaying* of language and logic. In 'The Intuitionist Foundation of Mathematics' (1931: 53), he wrote that the 'linguistic accompaniment is not a representation of mathematics; still less is it mathematics itself'. In the book *Intuitionism* (1956: 5), he echoes Brouwer's claim that language is an imperfect medium for communicating the real constructions of mathematics. The formal system is itself a legitimate mathematical construction, but 'one is never sure that the formal system represents fully any domain of mathematical thought; at any moment the discovering of new methods of reasoning may force us to extend the formal system'. Heyting claimed that 'logic is dependent on mathematics', not the other way around. So he did not intend his work in logic to *codify* intuitionistic reasoning. Nothing can do that.

Be this as it may, Heyting's formal work allowed intuitionistic (and constructivist) mathematics to come under the purview of ordinary proof theory, and there is now an extensive literature on formalized versions of intuitionistic arithmetic, analysis, set theory, and so on. Much (but not all) of the meta-theoretical work on intuitionistic logic employs a classical meta-theory. That is, the typical proof-theorist uses classical logic in order to study formal systems that themselves employ intuitionistic logic. One lasting contribution, at least from the point of view of the classical mathematician, has been a detailed study of the role of excluded middle in the practice of mathematics. We now know just how different intuitionistic mathematics is from classical mathematics—assuming (against Brouwer and Heyting) that intuitionistic formal systems accurately model intuitionistic mathematics. The same goes for Bishop's constructivism (see note 6 above). The meta-mathematical work has also led to a vigorous debate on the extent to which intuitionistic mathematics can serve the needs of science.[7]

Heyting's philosophical writing reiterates Brouwer's thesis that mathematics is mind-dependent and the focus on mathematical construction:

> The intuitionist mathematician proposes to do mathematics as a natural function of his intellect, as a free, vital activity of thought. For him, mathematics is a production of the human mind . . . [W]e do not attribute an existence independent of our thought, i.e., a transcendental existence, to . . . mathematical objects . . . [M]athematical objects are by their very nature dependent on human thought. Their existence is guaranteed only insofar as they can be determined by thought. They have properties only insofar as these can be discerned in them by thought . . . Faith in transcendental . . . existence must be rejected as a means of mathematical proof . . . [T]his is the reason for doubting the law of excluded middle. (Heyting 1931: 52–53)

[7] As we saw in the previous section, Brouwer would not care too much about the outcome of this debate.

With his teacher, Heyting argues that classical mathematics relies on a 'metaphysical' principle that the objects of mathematics exist independently of the mathematician and that the truths of mathematics are objective and eternal. He concedes that a mathematician is free to hold or reject such metaphysical principles in his spare time. However, the only way to avoid 'a maze of metaphysical difficulties' is to 'banish them from mathematics' itself (Heyting 1956: 3). Heyting accuses the classical mathematician of invoking metaphysical arguments via excluded middle:

If 'to exist' does not mean 'to be constructed', it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not. We have no objection against a mathematician privately admitting any metaphysical meaning he likes, but Brouwer's programme entails that we study mathematics as something simpler, more immediate than metaphysics. In the study of mental mathematical constructions 'to exist' must be synonymous with 'to be constructed'. (Heyting 1956: 2)

In short, Heyting insists that the practice of mathematics should not rely on any metaphysics.[8]

In some places, he seems to go further with the mind-dependence, and even to claim that mathematics is *empirical*:

A mathematical proposition expresses a certain expectation. For example, the proposition, 'Euler's constant $C$ is rational' expresses the expectation that we could find two integers $a$ and $b$ such that $C = a/b$ . . . The affirmation of a proposition means the fulfillment of an intention. The assertion '$C$ is rational', for example, would mean that one has in fact found the desired integers . . . The affirmation of a proposition is not itself a proposition; it is the determination of an empirical fact, viz., the fulfillment of the intention expressed by the proposition. (Heyting (1931: 59)

Intuitionistic mathematics consists . . . in mental constructions; a mathematical theorem expresses a purely empirical fact, namely the success of a certain construction. '2 + 2 = 3 + 1' must be read as an abbreviation for the statement: 'I have effected the mental constructions indicated by "2 + 2" and by "3 + 1" and I have found that they lead to the same result'

[8] In §5 of the previous chapter we saw that Haskell Curry claimed that a main virtue of his formalism is that it is free of metaphysical assumptions. Metaphysics-avoidance seems to be a common condition among philosophers of mathematics.

... [S]tatements made about the constructions ... express purely empirical results. (Heyting 1956: 8)

I suggest, however, that statements like these should not be taken too literally. Heyting was not advocating an empiricism like that of John Stuart Mill (see ch. 4, §3 above). Suppose that someone did a study of human beings doing sums. If '2 + 2' and '3 + 1' were replaced with seven-digit numbers, the empirical results would certainly differ from the mathematical ones. After all, humans do make mistakes. Surely, Heyting would take the empirical data to be irrelevant to mathematics. Along similar lines, the intuitionist accepts theorems like 'either $2^{1001} + 1$ is prime or $2^{1001} + 1$ is composite' even though the size of the factors (if any) would defy actual empirical realization.

We have encountered similar idealizations several times before in this study. I suggest that idealizations make it difficult for either party to claim that their view is the metaphysically neutral one. In philosophy of mathematics, metaphysics is all but inevitable—although one can query the relevance of metaphysics to the *practice* of mathematics (see ch. 1, §2). Brouwer's own Kantian position is not metaphysically neutral. He expresses definite views on the nature of mathematics and its entities. One would think that the best way to approach neutrality would be to reject Brouwer's free choice sequences and to stick with something more like Bishop's constructivism (note 6 above), the common core of classical and intuitionistic mathematics. Heyting (1931: 57) admits that intuitionistic mathematics would be 'impoverished' if free choice sequences were dropped. And classical mathematics would be impoverished without excluded middle.

Heyting's early paper (1931) reflects Brouwer's claim that classical mathematics is flawed and should be replaced with intuitionism: 'intuitionism is the only possible way to construct mathematics.' However, his book (1956) is more eclectic, arguing that intuitionistic mathematics deserves a place 'alongside' classical mathematics. Heyting wrote that the intuitionist does not claim a 'monopoly' on mathematics, and will rest content if the classical mathematician 'admits the good right of' the intuitionistic conception. A nice compromise. However, Heyting remained dubious of the 'metaphysical' assumptions that supposedly underlie classical mathematics.

## 4. Dummett

Recall that both Brouwer and Heyting considered language to be an imperfect medium for communicating mental mathematical construction, the real essence of mathematics. For them, logic concerns the mere forms for the deployment of this medium, and so a direct focus on language and logic is far removed from the proper field of debate. In contrast, Michael Dummett's main approach to mathematics and its logic is linguistic from the start. His philosophical interests lie more with intuitionistic logic than with mathematical matters (although free choice sequences are treated in Dummett 1977: ch. 3). Like Brouwer and unlike Heyting, Dummett does not have an eclectic orientation. Rather, he explores the thesis that 'classical mathematics employs forms of reasoning which are not valid on any legitimate way of construing mathematical statements . . .' (Dummett 1973: 97).

Dummett suggests that any consideration concerning which logic is correct must ultimately turn on questions of *meaning*. He thus adopts a widely held view that the rules for drawing inferences from a set of premisses flow from the meaning of some of the terms in the premisses, the so-called 'logical terminology'. This is consonant with the thesis that logical inference is analytic, or meaning-constitutive.

By its nature language is a public medium, and as such, the meanings of the terms in a language are determined by how the terms are correctly used in discourse. As Lewis Carroll's Humpty Dumpty might put it, the users of a language are in charge of how the terms are to be used. Their use determines meaning. What else can? Dummett (1973: 98–99) forcefully elaborates this point:

> The meaning of a mathematical statement determines and is exhaustively determined by its *use*. The meaning of such a statement cannot be, or cannot contain as an ingredient, anything which is not manifest in the use to be made of it, lying solely in the mind of the individual who apprehends that meaning . . . if two individuals agree completely about the use to be made of [a] statement, then they agree about its meaning. The reason is that the meaning of a statement consists solely in its rôle as an instrument of communication between individuals . . . An individual cannot communicate what he cannot be observed to communicate: if an

individual associated with a mathematical symbol or formula some mental content, where the association did not lie in the use he made of the symbol or formula, then he could not convey that content by means of the symbol or formula, for his audience would be unaware of the association and would have no means of becoming aware of it.

To suppose that there is an ingredient of meaning which transcends the use that is made of that which carries the meaning is to suppose that someone might have learned all that is directly taught when the language of a mathematical theory is taught to him, and might then behave in every way like someone who understood the language, and yet not actually understand it, or understand it only incorrectly.

I presume that the same goes for non-mathematical language as well.[9]

This common-sense view of language supports Dummett's *manifestation requirement*, a thesis that anyone who understands the meaning of an expression must be able to demonstrate that understanding through her behaviour—through her *use* of the expression: 'there must be an observable difference between the behaviour or capacities of someone who is said to have . . . knowledge [of the meaning of an expression] and someone who is said to lack it. Hence it follows . . . that a grasp of the meaning of a mathematical statement must, in general, consist of a capacity to use that statement in a certain way, or to respond in a certain way to its use by others.' Dummett identifies an important criterion of any semantics that is to play a role in philosophy: understanding should not be ineffable. One understands the expressions available in a language if, and only if, one knows how to use the language correctly.

The common slogan for such views is 'meaning is use', but this can be misleading. Advocates of the views are often criticized for leaving 'use' vague. Surely some account is needed if this notion is to have such a central place in philosophy. As Ludwig Wittgenstein

[9] Dummett's target includes Frege's view that the 'senses' of expressions are objective, mind-independent entities (e.g. Frege 1892). According to that view, to understand a sentence is to grasp its sense. Dummett (1973: 100) wrote that a 'notion of meaning so private to the individual is . . . irrelevant to mathematics as it is actually practised, namely as a body of theory on which many individuals are corporately engaged, an inquiry within which each can communicate his results to others'. Note the stark contrast with Dummett's fellow intuitionists Brouwer and Heyting.

(1978: 366–367) put it, 'It all depends [on] *what* settles the sense of a proposition. The use of the signs must settle it; but what do we count as the use?'

Some articulations of 'use' make it absurd to motivate the revision of logic and mathematical practice through considerations of meaning. If everything the mathematician does (and gets away with) is legitimate use, then the law of excluded middle is as legitimate as anything. As negation and disjunction are used in practice, excluded middle is correct. Practising mathematicians do not balk at its employment, and surely they know what they are talking about if anybody does. For better or worse, classical logic has won the day among mathematicians. So how can there be an argument for rejecting the law of excluded middle along semantic lines? On a view like this, it seems, *all* use is sacrosanct. Well, as Wittgenstein asked, 'what do we count as the use?'

There are at least two orientations toward mathematical language that would suggest an interpretation of 'use' along such strongly anti-revisionist lines. One such view is formalism, the thesis that correct mathematical practice can be codified into formal deductive systems (see the previous chapter). If classical logic is an ingredient of the appropriate deductive systems, then the issue of classical logic is settled. Suffice it to note that when both Dummett and the previous intuitionists—including Heyting—speak of 'proof', they do not mean 'proof in a fixed formal system'. For the intuitionist, proof is inherently informal. Formalism and intuitionism are not natural allies.

Another anti-revisionist understanding of language 'use' is what Dummett calls a 'holistic' account: 'On such a view it is illegitimate to ask after the content of any single statement . . . [T]he significance of each statement . . . is modified by the multiple connections it has . . . with other statements in other areas of language taken as a whole, and so there is no adequate way of understanding the statement short of knowing the entire language.' W. V. O. Quine's 'web of belief' is perhaps a view like this (see ch. 8, §2). Dummett argues that on such a semantic holism, there is no way to *criticize* a particular statement, such as an instance of the law of excluded middle, short of criticizing the entire language. This is not quite correct. Quine himself raises the possibility of changes to logic and mathematics owing to recalcitrant empirical data. Clearly, however, on a holistic view like Quine's, criticism of mathematical practice does not come from *semantics*, nor from reflections on meaning and understanding generally.

Dummett (1991a) suggests that the enterprise of semantic theory does not go well with the sort of semantic holism now under consideration. We need not adjudicate this here. A typical semantics is *compositional* in the sense that the semantic content of a compound statement is analysed in terms of the semantic content of its parts. In the prevailing Tarskian semantics, for example, the truth conditions of a complex formula are defined in terms of the truth conditions of its subformulas. For Dummett, the problem is that this semantics runs afoul of the manifestation requirement. On a classical, bivalent interpretation of a mathematical theory,

> the central notion is that of truth: a grasp of the meaning of a sentence . . . consists in a knowledge of what it is for that sentence to be true. Since, in general, the sentences of the language will not be ones whose truth-value we are capable of effectively deciding, the condition for the truth of such a sentence will be one which we are not, in general, capable of recognising as obtaining whenever it obtains, or of getting ourselves into a position in which we can so recognise it. (Dummett 1973: 105)

To satisfy the manifestation requirement, Dummett argues that *verifiability* or assertability should replace truth as the main constituent of a compositional semantics. Presumably, language users can manifest their understanding of the conditions under which each sentence can be verified or asserted. In mathematics, verification is *proof*, since a mathematician can assert a given sentence only if she has proved it. Dummett's proposal thus invokes the central theme of Heyting's semantics for intuitionistic logic. Instead of providing truth conditions of each formula, we supply proof conditions (see §3 above, or Dummett 1977: ch. 1 for details).

Dummett's alternative to semantic holism is what he calls a *molecular* semantics, according to which: 'individual sentences carry a content which belongs to them in accordance with the way they are compounded out of their own constituents, *independent of other sentences of the language not involving those constituents* . . .' (Dummett 1973: 104). Dummett's proposal is that at least some crucial parts of language can be understood independently of any other parts. This applies, first and foremost, to the logical terminology: connectives such as negation, conjunction, disjunction, and 'if–then', and quantifiers like 'there is' and 'for all'. Neil Tennant (1997: 315), a prominent Dummettian, puts it well:

the contention here is that the analytic project must take the [logical] operators one-by-one. The basic rules that determine logical competence must specify the unique contribution that each operator can make to the meanings of complex sentences in which it occurs, and, derivatively, to the validity of arguments in which such sentences occur . . . It follows . . . that one [should] be able to master various fragments of the language in isolation, or one at a time. It should not matter in what order one learns (acquires grasp of) the logical operators. It should not matter if indeed some operators are not yet within one's grasp. All that matters is that one's grasp of any operator should be total simply on the basis of schematic rules governing inferences involving it.

On a view like this, established practice can be criticized. An analysis of language might reveal an incoherence in how the logical operators are used. In particular, the philosopher might discover a disharmony between different aspects of how the terms are used. Dummett and, with more detail, Tennant argue that the ways that logical operators are typically introduced into proofs conflicts with classical principles and inferences. That is, the rules for introducing—and showing that one grasps the meaning of—the negation and disjunction operators separately do not justify excluded middle when the connectives are combined. Tennant (1997: 317) calls excluded middle a 'shoddy marriage of convenience'. Thus, Dummett and Tennant support Heyting's argument that intuitionistic logic is justified on this semantics, but classical logic is not.

In the Dummettian framework, a major presupposition of classical mathematics is that there are, or may be, truths that cannot become known. A bivalent semantics suggests that truth is one thing and knowability another. Dummett's approach, sometimes called *global semantic anti-realism*, entails that, at least in principle, all truths are knowable. The possibility of an unknowable truth is ruled out a priori. As we saw in §2 above, Brouwer himself adopted semantic anti-realism for mathematics, as well as 'practical life and science'.

The going here is not straightforward. Notice that even with Heyting semantics, a language satisfies the manifestation requirement only under the pesky idealizations encountered earlier. No one can manifest understanding of the proof conditions of a long formula, and no one can know of some large numbers whether they are prime. Again it is a theorem of intuitionistic arithmetic that every natural number is either prime or composite. The reason is that we have a finite *method* for determining whether a number is prime. It does not matter how feasible this method is or even whether anyone has carried it out in a given case. In particular, $2^{1001} + 1$ is either prime or composite. Tennant (1997: ch. 5) provides a lucid defence of the idealizations needed to support Heyting semantics.

On the other hand, we should not idealize too much. The route from Heyting semantics to the repudiation of classical logic depends on a certain pessimism concerning human mathematical abilities (see, for example, Posy 1984 and Shapiro 1991: ch. 6). If human beings are capable of deciding the truth value of every well-formed mathematical statement, then classical logic will prevail after all—even under Heyting semantics. It seems that Dummett's intuitionism lies between a strict finitistic view that we only understand what we have actually proved, and either a straightforward realism that countenances unknowable truths or a robust optimism that holds that for each unambiguous mathematical sentence $\Phi$, the mathematician can determine whether $\Phi$ is true or false. Tennant (1997: chs. 6–8) provides a defence of intuitionistic logic as the right balance of these possibilities.

A defender of classical logic has two options in light of Dummett's critique. One is to provide a semantics that meets the manifestation, separability, and harmony requirements and still sanctions classical logic. A philosopher who takes this route accepts the broadly Dummettian framework and, working within that framework, argues that classical logic is justified. The debate is likely to turn on questions of semantics, proper idealization, and the extent and details of manifestation. Another option for the classical mathematician would be to reject Dummett's entire framework. The philosopher concedes that classical logic does not enjoy the kind of justification that Dummett demands, but she argues that classical mathematics does not stand in need of this sort of justification. The fruit and power of classical mathematics establishes its place in our intellectual lives. If classical mathematics conflicts with the Dummettian framework for semantics, it is the latter that must go. Those who lean this way may be tempted by holism (see ch. 8, §2).

As Dummett (1973: 109) himself points out, the foregoing considerations are very general, turning solely on how language is understood. Thus, if his conclusions are sound they support the adoption of intuitionistic logic for all discourse, not just for mathematics. So Dummett goes beyond the prior intuitionists Brouwer and Heyting, who agree that classical logic is appropriate for ordinary reasoning about finite collections of mind-independent objects. This motivates Dummett to search for other arguments for intuitionistic logic that depend only on features special to mathematics: 'Is there, then, any alternative defence of the rejection, for mathematics, of classical in favour of intuitionistic logic? Is there any such defence which turns on the fact that we are dealing with *mathematical* statements in particular, and leaves it open whether or not we wish to extend the argument to statements of any other general class?'

Dummett concludes that one route to such a 'local' revisionism is a 'hard-headed' finitism in which one denies that there is a determinate fact concerning the outcome of a procedure that has not been carried out. On this view, one cannot conclude

$$2^{1001} + 1 \text{ is prime or composite}$$

until one has carried out the relevant procedure. So one demurs from the idealizations discussed previously. On such a view excluded middle remains unjustified, but the hard-headed finitist has to restrict logic even further than the intuitionist does. It is not clear which inferences and principles of intuitionistic logic are justified from the hard-headed approach. Some of the (intuitionistically correct) inferences that lead to '$2^{1001} + 1$ is prime or composite' have to be jettisoned. If the intuitionist does not throw out the baby with the dirty bathwater, surely the hard-headed finitist does.

Dummett's recent work (1991, 1991a) provides another angle on the repudiation of classical mathematics. In an early paper on Gödel's incompleteness theorem (Dummett 1963), he defines a concept to be *indefinitely extensible* if it is not possible to delineate the range of objects to which the concept applies. That is, a concept is indefinitely extensible if any attempt to delineate the extension of the concept leads to an instance of the concept not so delineated. Dummett argues that the incompleteness theorem shows that the notion of arithmetic truth is indefinitely extensible. Let *T* be any effective procedure for enumerating arithmetic truths. An application of the incompleteness theorem yields an arithmetic truth $\Phi$ not enumerated by $T$. So $T$ fails as a characterization of arithmetic truth.

Dummett argues that virtually any substantial mathematical domain—the natural numbers, the real numbers, the set-theoretic hierarchy, and so on—is indefinitely extensible. Any attempt to delimit the domain leads to extensions of it. This may be related to the prior intuitionistic claim that there is no actual infinity, only potential infinity.

Let $d$ be a domain. Dummett suggests that a quantifier 'for every $d$' is coherent whether or not $d$ is indefinitely extensible. Otherwise, mathematics is doomed from the start. In the later work, however, he argues that classical logic applies to a domain only if it is not indefinitely extensible. This conclusion is based in part on an analysis of mathematical logic, model theory in particular. In providing an interpretation of a formal language, one is required to specify a *domain of discourse*. Dummett argues that the usual proof that classical logic is sound for classical model theory presupposes that such a domain is definite—not indefinitely extensible. Thus, he argues, classical logical theory does not apply to mathematics where the range of the quantifiers is indefinitely extensible. But we still need an argument that full intuitionistic logic applies to such domains.

## 5. Further Reading

Benacerraf and Putnam 1983 contains a delightful dialogue from Heyting 1956, and translations of Brouwer 1912 and 1948, and Heyting 1931. It also has Dummett 1973. Van Heijenoort 1967 contains translations of other relevant papers by Brouwer, notably Brouwer 1923. Another interesting source in English is Brouwer 1952. Heyting 1956 and Dummett 1977 are excellent book-length introductions to intuitionistic mathematics, both in English. Dummett 1978 contains many of his important philosophical papers (including 1963 and 1973). For more on indefinite extensibility, see Dummett 1994. Dummett's more recent work in the philosophy of mathematics is developed in 1991 and 1991a. See Tennant 1987 and 1997 for an extensive development of a broadly Dummettian programme. See also Prawitz 1977.