3

PLATO'S RATIONALISM, AND ARISTOTLE

> Let's start at the very beginning. A very good place to start.
>
> (*The Sound of Music*)

It is natural to begin our historical sketch in ancient Greece, since it is widely agreed that both mathematics and philosophy, as we know them today, were born there. Apparently, pre-Greek mathematics consisted mainly of calculation techniques and numeration systems, concerned with either religion or practical matters like dividing land. For better or worse, the Greek mathematicians introduced the focus on exactitude and rigorous proof.

Legend has it that the oracle of Apollo once said that a plague would end if a certain altar were doubled in size, maintaining its shape. If the concerned citizens had increased each dimension of the altar by a third, the result would be an object about 2.37 times its original size. One would think that the god would be pleased with this additional 37%, but the legend is that the plague continued after they doubled each side of the altar, increasing its size *eightfold*. If the citizens increased the original sides by 26%, the altar would be about 2.0004 times its original volume. Surely, that would please the god. The difference between twice the size and 2.0004 times the size is not detectable experimentally, at least by humans. However, the Greek mathematicians took the task as one of doubling the altar *exactly*. They were not interested in an approximation, no matter how close it may be. This 'practical' issue of averting disaster supposedly led to the geometrical problem of doubling the cube: given a line segment, and using only a compass and unmarked straight edge, to produce a line segment whose cube is exactly double that of the original. The mathematicians wanted it exact and they wanted it proved. Two similar problems were to trisect an angle and to produce a line segment whose square has the same area as that of a given circle. Arbitrarily close approximations were available, but did not count. These problems occupied mathematicians for centuries, culminating more than 2,000 years later with the result that there are no solutions—the tasks are impossible.

Thomas Kuhn's influential *Structure of Scientific Revolutions* (1970) speaks of revolutions and 'paradigm shifts' that make it difficult to understand scientific works of the past. According to Kuhn, to understand previous work we have to unlearn our current science and try to immerse ourselves in the overturned world-view. Intervening revolutions have forever changed the concepts and tools of the day, making the past work 'incommensurable' with ours. What of mathematics? If Kuhn's philosophy and historiography of science apply to mathematics, the revolutions and paradigm changes are far more subtle. A contemporary mathematician does not have to do much (if any) conceptual retooling in order to read and admire Euclid's *Elements*. Modern logical techniques have uncovered a few gaps in the reasoning, but Euclid's concerns look like ours, and so do his proofs and constructions. The logical gaps notwithstanding, the *Elements* are a model of mathematical rigour. It is widely believed that the *Elements* are a culmination of a research programme that was well under way during Plato's lifetime.

Ancient Greece was also the birthplace of western, secular philosophy. We see Socrates, Plato, and Aristotle (as well as some of the pre-Socratic philosophers) struggling with many of the issues that concern today's philosophers, including some of the issues treated in the present book. Plato stands at the head of a long tradition in philosophy sometimes called *rationalism* or 'Platonism' (or 'platonism', if one wants a little distance from the master). The next section is a brief account of Plato's general philosophy, or theory of Forms. This is followed by a discussion of Plato's views on mathematics—arithmetic and geometry in particular. The succeeding section reverses the orientation, and deals with the influence of mathematics on Plato's philosophical development. The final section of this chapter is on Aristotle, Plato's pupil and main opponent. It serves as a transition into the treatment of empiricism later in the book (e.g. ch. 4, §3; ch. 8, §2).

1. The World of Being

Plato was motivated by a gap between the ideas we can conceive and the physical world around us. For example, although we have tolerably clear mental pictures of justice, everything we see and hear falls short of perfect justice. We have a vision of beauty and yet nothing is completely beautiful. Nothing is completely pious, virtuous, and so on. Everything in the material world has flaws. Of course, Socratic questioning would surely reveal that our conceptions of justice, beauty, and the like are not as clear as they sometimes seem to be, but this does not detract from the present observations concerning defects in the physical realm. We have *some* understanding of the perfect ideals, and yet we never find them. Why is this?

Plato's answer is that there is a realm of Forms, which contains perfect items like Beauty, Justice, and Piety. He sometimes speaks of 'Beauty itself', 'Justice itself', and 'Piety itself'. A physical object, such as a painting, is beautiful to the extent that it 'resembles', 'participates in', or 'has a share of' Beauty itself. A person is just to the extent that she resembles Justice itself. Plato calls the physical realm the world of Becoming, because physical objects are subject to change and corruption. They get better and they get worse. What is beautiful can become ugly. What is virtuous can become vicious. In contrast, the Forms are eternal and unchanging. Beauty itself was, is, and always will be the same; individual things are beautiful to the extent that they conform to this timeless, unchanging standard. Clearly, then, Plato would not subscribe to the slogan that beauty is in the eye of the beholder. The same goes for justice and the other Forms. There is nothing subjective, or conventional, or culture-relative about them.

That, in short, is Plato's ontology of Forms. What of his epistemology? How do we know about, or apprehend these Forms? We understand the physical world—the world of Becoming—through the senses. He calls this the realm of 'sights and sounds'. In contrast, we grasp the Forms only through mental reflection. We see and hear beautiful things and just people, but we have to *think* our way to Beauty and Justice. The following passage from Book 6 of the *Republic* is typical:

> Let me remind you of the distinction we drew earlier and have often drawn on other occasions, between the multiplicity of things that we call good or beautiful or whatever it may be and, on the other hand, Goodness itself or Beauty itself and so on. Corresponding to each of these sets of many things, we postulate a single Form or real essence as we call it . . . Further, the many things, we say, can be seen, but are not objects of rational thought; whereas the Forms are objects of thought, but invisible.

The *Meno* suggests another epistemology. There, Plato has Socrates lead a slave to the theorem that the square on the diagonal of a given square is double the area of the original square. Socrates emphasizes that neither he, nor anyone else, taught the theorem to the slave. By asking carefully chosen questions, and pointing to aspects of a drawn diagram, Socrates gets the slave to discover the theorem for himself. Plato uses the experiment to support a doctrine that when it comes to geometry—or the world of Being generally—what is called 'learning' is actually *remembering* from a past life, presumably a time when the soul had direct access to the world of Being.

Scholars disagree on the nature and role of this 'recollection' in Plato's epistemology, and most subsequent Platonists demur from it. In any case, Plato did hold that the soul is in a third ontological category, with the ability to apprehend both the world of Being and the world of Becoming.

With or without the 'mystical' elements of the epistemology, one gets the impression from the dialogues that the physical world is constructed as it is just so that we will be driven beyond our senses to investigate the world of Being. For Plato, mathematics is a key step in this process. It elevates the soul, reaching beyond the material world to the eternal world of Being.

2. Plato on Mathematics

Mathematics, or at least geometry, provides a straightforward instance of the gap between the flawed material world around us and the serene, ideal, perfect world of thought. From before Plato's time until today we have had completely rigorous definitions of straight line, circle, and so on, but the physical world contains no perfectly straight lines without breadth, and no perfect circles, or at least none that we can see. Perhaps breadthless straight lines and perfect circles, and the like, are part of the physical space (or space-time) that we all occupy, but even so, we do not encounter them, as such, in any physical way. So what do we study in geometry, and how do we study it?

To labour the obvious, Plato believed that the propositions of geometry are objectively true or false, independent of the mind, language, and so on of mathematicians. In the terminology of Chapter 2, he was a realist in truth-value. This realism is more or less assumed, and not defended, throughout the dialogues. Perhaps there were no serious alternatives. But what is geometry about? What is its ontology? How is geometry known? Plato held that the subject-matter of geometry is a realm of objects that exist independent of the human mind, language, and so on. He argued from realism in truth-value to realism in ontology, a theme echoed throughout subsequent history. Plato's main contentious claims concern the *nature* of geometrical objects and the *source* of geometrical knowledge. He believed that geometrical objects are not physical, and that they are eternal and unchanging. In this sense, at least, geometrical objects are like Forms and are in the world of Being. He would thus reject the above suggestion that geometric objects exist in physical space.

At the end of Book 6 of the *Republic* Plato gives a metaphor of a divided line (see Fig. 3.1). The world of Becoming is on the bottom and the world of Being on the top (with the Form of Good on top of everything). Each part of the line is again divided. The world of Becoming is divided into the realm of physical objects on top and reflections of those (e.g. in water) on the bottom. The world of Being is divided into the Forms on top and the objects of mathematics on the bottom.[1] This suggests that physical objects

[1] The divisions are unequal, with the Forms getting the largest space. The following double proportion holds: Forms are to mathematical objects as physical objects are to reflections, as Being (i.e. Forms plus mathematical objects) is to Becoming (i.e., physical objects and reflections). Although Plato does not mention this, it follows that the 'mathematical objects' segment is exactly the same size as the 'physical objects' segment.

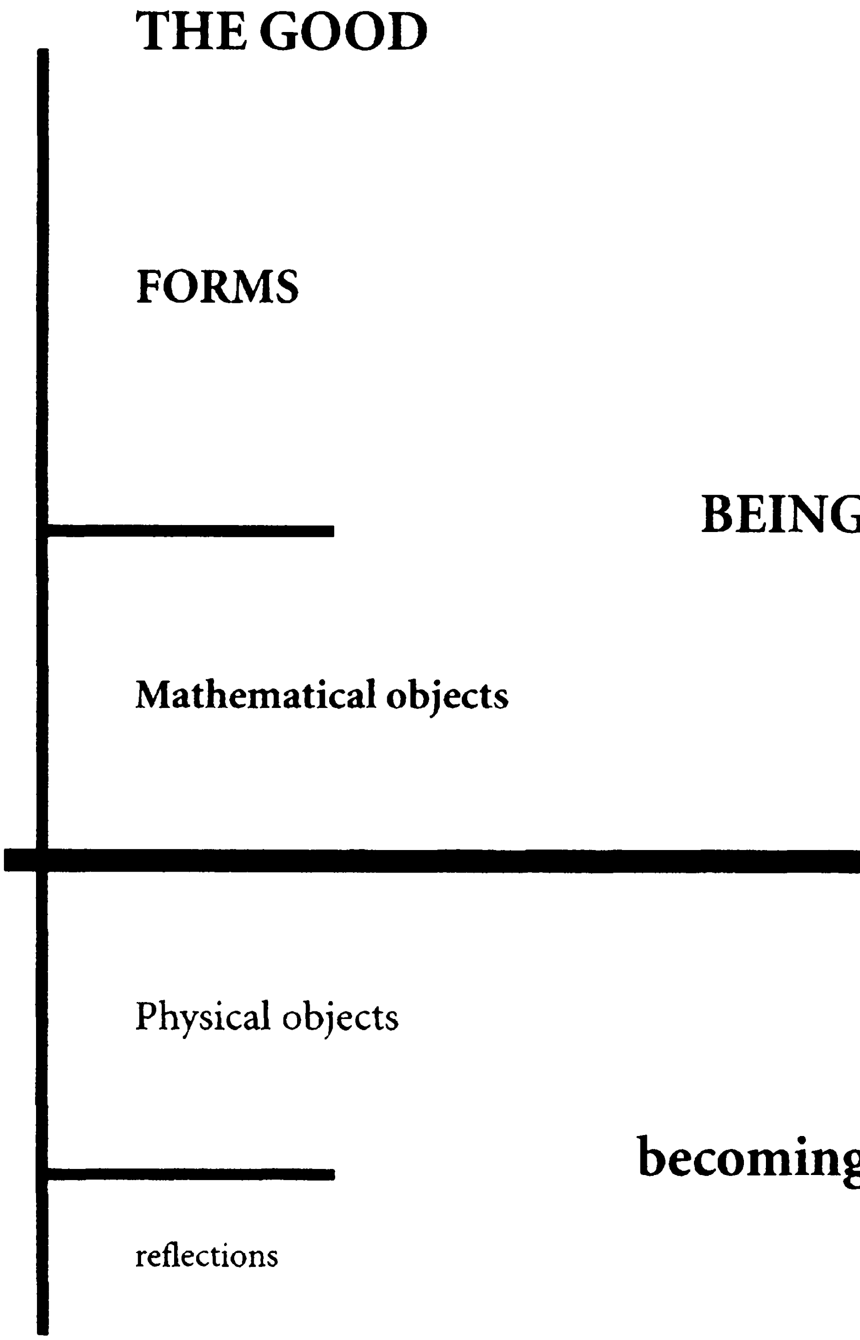

Fig. 3.1. The divided line

are 'reflections' of mathematical objects which, in turn, are 'reflections' of Forms.

However, there is evidence, including some attributions of Aristotle, that Plato took at least some mathematical objects to be Forms. There are hints that during his later neo-Pythagorean period Plato took all Forms to be mathematical. There are accounts of a public lecture on the Good, where, to the disappointment of some of his audience, Plato spoke almost exclusively of mathematical matters.

We need not settle these exegetical details. A common thread, of all periods and all interpretations, is that Plato's world of geometry is divorced from the physical world and, more important, geometrical knowledge is divorced from sensory observation. Geometric knowledge is obtained by pure thought, or by remembering our past acquaintance with the geometric realm, as above.

Concerning ontology, and at least the negative side of epistemology, Plato's argument is deceptively simple. The propositions of geometry concern points that have no dimensions, perfectly straight lines that have no breadth, and perfect circles. The physical world contains no such items, and we do not see Euclidean points, lines, and circles. Thus, geometry is not about anything in the physical world, the world of Becoming, and we do not apprehend geometric objects via the senses. Of course, some physical objects *approximate* Euclidean figures. The circumference of an orange and a carefully drawn circle on paper more or less resemble Euclidean circles, the orange less, the drawn circle more. But geometric theorems do not apply to these approximations. Consider, for example, the theorem that a tangent to a circle intersects the circle at a single point. Even if one carefully draws a circle and a tangent straight line, using fancy, expensive tools or a very sharp pencil (or high-resolution printer), one will still see that the line overlaps the boundary of the circle in a small region, not a single point (see Fig. 3.2). If one uses a chalk-board or a stick in sand for the exercise, the overlap will be considerably larger. Of course, none of this

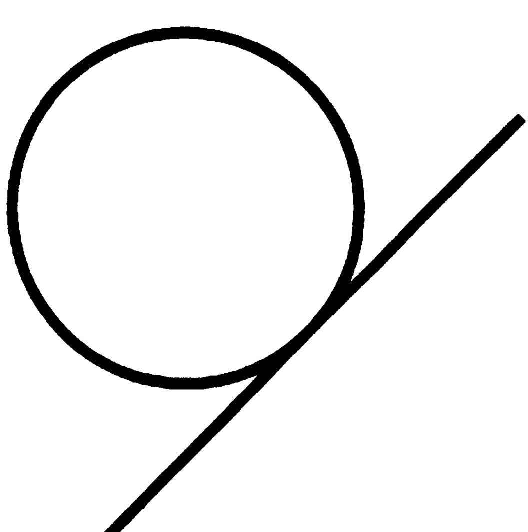

Fig. 3.2. Tangent to circle

disconfirms the standard theorem that the intersection of a circle and a tangent is a single point. Plato's explanation is straightforward. The drawn circles and lines are only poor approximations of the real Circle and the real Line, which we grasp only with the mind (or remember). The small boundary overlapping the drawn figures is a poor approximation of a point.

We are in position to better understand Plato's remark in the passage from Book 7 of the *Republic*, quoted in chapter 1:

> [The] science [of geometry] is in direct contradiction with the language employed by its adepts . . . Their language is most ludicrous . . . for they speak as if they were doing something and as if all their words were directed toward action . . . [They talk] of squaring and applying and adding and the like . . . whereas in fact the real object of the entire subject is . . . knowledge . . . of what eternally exists, not of anything that comes to be this or that at some time and ceases to be. (Plato, 1961, 527a in the standard numbering)

If Plato is correct that geometry concerns eternal and unchanging items in the world of Being, then there should be no dynamic language in geometry. It is hard for a Platonist to make sense of the constructions in Euclid's *Elements*, for example. According to the fifth-century neoplatonist Proclus (1970), the problem of 'how we can introduce motion into immovable geometric objects' occupied many of the best minds at Plato's Academy for generations after.

There is a similar issue concerning the diagrams that usually accompany geometric demonstrations. A Platonist would surely worry that these might confuse the reader into thinking that the theorem is about the physically drawn diagram. What, after all, is the purpose of the diagrams? Plato's explanation might be that the diagram somehow aids the mind in grasping the eternal, unchanging geometric realm, or helps us to recall the world of Being. However, one might wonder how this is possible, since the world of Being is not accessible via the senses. In the *Republic* (510d), Plato writes:

> You . . . know how [geometers] make use of visible figures and discourse about them, though what they really have in mind is the originals of which these figures are images. They are not reasoning, for instance, about this particular square and diagonal which they have drawn, but about the Square and the Diagonal; and so in all cases. The diagrams they draw and

the models they make are actual things, which may have their shadows or images in water; but now they serve in their turn as images, while the student is seeking to behold those realities which only thought can comprehend.

Here we have the same metaphor as in the divided line: reflections and images. I suppose the advanced mathematician would have no need for diagrams, being in more direct touch with the geometric universe. Plato was not the last philosopher to wonder about the role of diagrams in geometric demonstration.

Although, as noted, subsequent Platonists did not adopt the more mystical aspects of Plato's epistemology, most of them maintained that geometrical knowledge is a priori, independent of sensory experience. It may be that some sensory experience is necessary to grasp the relevant concepts, or we may need drawn diagrams as a visual aid to the mind, or perhaps to awaken our minds to the eternal and unchanging geometric realm of Euclidean space. However, it is crucial that mathematical knowledge is in principle independent of sensory experience. The main reason for this comes from the Platonist ontology. Geometry is not about physical objects in physical space.

This view leaves a problem of explaining why geometry applies to the physical world, even approximately. In the *Timaeus* Plato provides a detailed, but speculative story of how the physical world was constructed geometrically, from the five so-called Platonic solids: tetrahedron (pyramid), octahedron, hexahedron (cube), icosahedron, dodecahedron.

The details of Plato's views concerning arithmetic and algebra are not as straightforward as his account of geometry, but the overall picture is the same. He was a straightforward realist in both truth-value and ontology, holding that propositions from arithmetic and algebra are true or false independent of the mathematician, the physical world, and even the mind, and he held that arithmetic propositions are about a realm of abstract objects called 'numbers'. In the *Sophist* (238a), the Stranger says that 'among the things that exist we include number in general', and Theaetetus replies, 'Yes, number must exist if anything does'.

The dialogues contain several passages that apply the Platonic distinctions to numbers. There are, of course, numbers of material objects, which we may call 'physical numbers'. This is number in the world of Becoming. These are distinguished from 'the numbers themselves', which are not grasped by the senses, but by pure thought alone.

In the *Philebus* (56), for example, Plato has Socrates distinguish between 'the ordinary man' and 'the philosopher' when it comes to arithmetic. There are, in a sense, two different arithmetics. The interlocutor, Protarchus, asks 'on what principle . . . is this distinction . . . to be based?' Socrates replies: 'The ordinary arithmetician, surely, operates with unequal units; his "two" may be two armies or two cows or two anythings from the smallest thing in the world to the largest, while the philosopher will have nothing to do with him, unless he consents to make every single instance of his unit precisely equal to every other of its infinite number of instances.' See also *Theaetetus*, 196, *Republic*, 525. We thus see that arithmetic, like geometry, applies to the material world only approximately, or only to the extent that objects can be distinguished from each other. The philosopher's arithmetic applies precisely and strictly only to the world of Being.

There is no consensus on Plato's opinions concerning the nature of number. One interpretation has it that Plato took numbers to be ratios of geometric magnitudes.[2] The number four, for example, would be the ratio of the perimeter of a square to one of its sides and also the ratio of the area of a square to the area of a square whose side is half the original. This approach has the advantage of covering not only natural numbers, but also (positive) rational and irrational numbers (as discussed in dialogues like *Theaetetus*). The disadvantage of this interpretation is that it does not account for the use of numbers in contexts other than geometry. Even if we restrict our focus to the world of Being, we count things other than geometric magnitudes. We say, for example, that a given equation has two roots, that there are five Platonic solids, and that there are four prime numbers less than ten.

The above passage from the *Philebus* suggests another account of Plato's arithmetic. When the ordinary arithmetician counts a pair of shoes, each shoe is a unit, but the two shoes are not the same shape or even exactly the same size. In contrast, when the philosopher counts 'two', she refers to a pair of units that are the same

[2] This, in effect, is how Euclid proceeded in the *Elements*, Book 10. Euclidean arithmetic is a branch of geometry.

in every way. For the philosopher, natural numbers are collections of pure units, which are indistinguishable from one another (*Republic*, 425; *Sophist*, 245).

Notice, incidentally, that for both the ordinary person and the philosopher, 'number' is always a number *of* something or other. The plain person's numbers are numbers of collections like armies and cows. The philosopher's numbers are numbers of pure units.

Several ancient sources distinguish the theory of numbers, called 'arithmetic' from the theory of calculation, called 'logistic'. Most writers take the latter to be a practical discipline, concerning measurement and business dealings (e.g. Proclus 1970: 20). One would think that this distinction would suit Plato well, given his stark contrast between the world of Being and the world of Becoming. Arithmetic would concern Being, while logistic would concern Becoming. However, Plato has both arithmetic and logistic focused on the world of Being. The difference concerns how the natural numbers themselves are to be studied. Arithmetic 'deals with the even and the odd, with reference to how much each happens to be' (*Gorgias*, 451). If 'one becomes perfect in the arithmetical art', then 'he knows also all of the numbers' (*Theaetetus*, 198). Plato's logistic differs from arithmetic 'in so far as it studies the even and the odd with respect to the multitude they make both with themselves and with each other' (*Gorgias*, 451). Arithmetic thus deals with the natural numbers individually and logistic concerns the relations among the numbers. For logistic, Plato proposed principles for how natural numbers are 'generated' from other natural numbers (through the gnomon). This is something akin to an axiomatic treatment of the genesis of the ontology.

Plato said that one should pursue both arithmetic and logistic for the sake of knowing. It is through the study of the numbers themselves, and the relations among numbers, that the soul is able to grasp the *nature* of numbers as they are in themselves. As Jacob Klein (1968: 23) put it, theoretical logistic 'raises to an explicit science that knowledge of relations among numbers which . . . precedes, and indeed must precede, all calculation'. Plato's logistic is to practical calculation as his geometry is to figures drawn on paper or sand.

One might wonder, with Klein (1968: 20), just what is to be studied in Plato's arithmetic, as opposed to his logistic. Presumably, the art of counting—reciting the numerals—is arithmetic *par* *excellence*. Yet 'addition and also subtraction are only an extension of counting'. Moreover, 'counting itself already presupposes a continual relating and distinguishing of the numbered things as well as of the numbers'. Klein (1968: 24) tentatively concludes that logistic concerns *ratios* among pure units, while arithmetic concerns counting, addition, and subtraction. In line with the later dialogues, it might be better to think of Plato's logistic as what we would call 'arithmetic', namely the mathematical study of the natural numbers. Plato's arithmetic is a part of higher philosophy, where one comes to grasp the metaphysical nature of number itself.

3. Mathematics on Plato

Plato's admiration for the exciting accomplishments of mathematicians is abundantly clear, even to a casual reader of the dialogues. As Gregory Vlastos (1991: 107) put it, Plato 'was able to associate in the Academy on easy terms with the finest mathematicians of his time, sharing and abetting their enthusiasm for their work'. Some recent scholars have focused attention on the influence of the development of mathematics on Plato's philosophy. In a dramatic way, light is cast on some of the sharp contrasts between Plato and his teacher Socrates.

As far as we know, Socrates' main interests were in ethics and politics, not mathematics and science. He considered himself to have a divine mandate to spread philosophy to everyone. We all delight in the image of Socrates roaming the streets of Athens discussing justice and virtue with anyone who would listen and talk. Anyone. He lived the slogan that philosophical reflection is the essence of living. We were born to think. At his trial, Socrates declared it would be disobedience to God for him to shut up and mind his own business (*Apology*, 38a): 'I tell you that to let no day pass without discussing goodness and all the other subjects about which you hear me talking and examining both myself and others is really the best thing a man can do, and that life without this sort of examination is not worth living.'

Socrates typically proceeds by eliciting the beliefs of an interlocutor and then, through careful questioning, attempts to draw out surprising and unwanted consequences of those beliefs. In most cases the encounter does not end with the *reductio ad absurdum* of the interlocutor's original position. Instead, the interlocutor is challenged to re-examine his beliefs and to learn by formulating new ones. Socrates even pursues this at his own trial, against his accusers.[3]

Socratic method, then, is a technique for weeding out false beliefs. If the method does produce truth, it is only by a process of elimination or perhaps trial and error. Socrates never claimed any special positive knowledge of justice, virtue, and so on. Quite the contrary. He took his wisdom to consist in the fact that he knows that he does not know. He probably arrived at this negative conclusion by examining himself.

Moreover, Socratic method does not result in certainty. It might inform us that some of our beliefs are false or confused, but it does not inevitably point to *which* of the beliefs are false or confused. The method is fallible and hypothetical, but it is the best we have.

The methodology of the mature Plato does not resemble that of Socrates in any of these ways. Plato notes in passing that mathematics is 'universally useful in all crafts and in every form of knowledge and intellectual operation—the first thing everyone has to learn' (*Republic*, 523).[4] By Plato's day one needed intense and prolonged study to master mathematics. A casual acquaintance with it would not get you very far. Thus, Plato realized that one needs intense and prolonged study for any 'form of knowledge and intellectual operation'. Especially philosophy.

Unlike his teacher, Plato held that philosophy is not for everyone. In the Commonwealth envisioned in the *Republic* only a few carefully selected leaders engage in philosophical reflection, and only after a training period lasting until they are at least 50 years old. The vast majority of the inhabitants are admonished to get their direction from these leaders and to mind their own business. Farmers stick to farming, and cooks stick to cooking. Everyone does only what he or she does best. Philosophy too is left to the experts—the Guardians. Plato even held that it is *dangerous* for the masses to

[3] Had the accusers, or the jury, realized the absurdity of their underlying assumptions, Socrates' life would have been spared. But all too often, trials are not won or lost on the basis of sweet reason.

[4] As noted in chapter 1, it is not a great exaggeration to say that this holds today as well. Consider the wide range of mathematics prerequisites throughout the natural and social sciences.

engage in philosophy. It is even dangerous for prospective Guardians to engage in philosophy before they have been properly trained. Plato insisted that for the vast majority of people the unexamined life is well worth living. If Plato had his way, the examined life would be forbidden to almost everyone. In this regard, it is harder to imagine a sharper contrast than that between Socrates and his most celebrated pupil.

It is noteworthy that for Plato, a full decade of the Guardians' training is devoted to mathematics. They do little else between the ages of 20 and 30. This is more than we expect from prospective professional mathematicians today. Plato's reason for this is clear. To rule well, the Guardians need to turn their focus from the world of Becoming to the world of Being. Thus, a crucial part of their education has to 'turn the soul from a day that is as dark as night to the true day, that journey up to the veritable world which we shall call the true pursuit of wisdom' (*Republic*, 521). Mathematics 'draws the soul from the world of change to reality'. It 'naturally awakens the power of thought . . . to draw us towards reality'—at least for the few souls capable of such ascent.

Plato's break with his teacher is understandable, if not admirable. Socrates did not give mathematics pride of place, while Plato saw mathematics as the gateway into the world of Being, a gateway that must be passed if one is to have any hope of understanding anything real.[5] Mathematics, the prerequisite to philosophical study, demands a long period of intense study. No wonder that most of us have to live our lives in ignorance of true reality, and must rely on Guardians for direction as to how to live well.

Plato's fascination with mathematics may also be responsible for his distaste with the hypothetical and fallible Socratic methodology. Mathematics proceeds (or ought to proceed) via *proof*, not mere trial and error. As Plato matures, Socratic method is gradually supplanted. In the *Meno* Plato uses geometric knowledge, and geometric demonstration, as the paradigm for all knowledge, including moral knowledge and metaphysics. In that dialogue Plato wants to make a point about *ethics*, and our knowledge of ethics, and he explicitly draws an analogy with geometrical knowledge. It is a standard Socratic and Platonic strategy to start with clear instances

[5] Recall the sign at the entrance to the Academy: 'Let no one ignorant of geometry enter here.'

and proceed to more-problematic cases, by way of analogy. Plato finds things clear and straightforward when it comes to mathematics and mathematical knowledge, and he tries to extend the findings there to all of knowledge. In the dialogue no one questions the analogy between mathematics and ethics or metaphysics. Rationalism is based on the same analogy (see ch. 4, §1).

During their ten years of mathematical study the prospective Guardians proceed 'hypothetically', from postulates and axioms. They must simply accept those 'hypotheses', and do not know what their ultimate foundation is. As indicated by the metaphor of the divided line, the mathematicians also use diagrams and other aids from the world of Becoming. At this stage the future Guardians proceed from the world of Becoming to the world of Being. This stage is necessary, but it is not a suitable conclusion to their studies. Plato hinted at a more certain and secure methodology for philosophy. Beginning at the age of 30—after the decade of mathematics—the prospective leaders spend some years engaged in 'dialectic', where they encounter and grasp the Forms themselves, independent of any soiled instances in the material world, and they arrive at unhypothetical first principles, the ultimate basis for all knowledge and understanding. The best among them will then ascend to contemplate the Good.

In sum, then, for Plato the fumbling but exciting and egalitarian Socratic method first gives way to the elite rigour of Greek mathematical demonstration. This is then replaced with an even more elite 'dialectical' encounter with the Forms.

4. Aristotle, the Worthy Opponent

Most of what Aristotle says about mathematics is a polemic against Plato's views, and there is not much consensus among scholars on the scattered positive remarks he makes. Nevertheless, there is at least the main direction of an account (or accounts) of mathematics that foreshadows some modern thinkers. Aristotle's philosophy contains seeds of empiricism.

As noted above, Plato's philosophy of mathematics is tied to his account of Forms as eternal, unchanging entities in the separate realm of Being. In like manner, Aristotle's philosophy of mathematics is tied to his *rejection* of a separate world of Being. Aristotle did accept the existence of Forms, or universals, but he held that they are not separate from the individual objects of which they are Forms. Beauty, for example, is what all beautiful things have in common, and not something over and above those beautiful things. If someone manages to destroy all beautiful things, she will destroy Beauty itself—for there will be nothing left for Beauty to exist in. The same goes for Justice, Virtue, Man, and the other Forms. In short, for Aristotle things in the physical world have Forms, but there is no separate world to house these Forms. Forms exist in the individual objects.

Aristotle sometimes suggests that the important question concerns the *nature* of mathematical objects, not their mere existence or non-existence: 'If mathematical objects exist, they must exist in perceptible objects as some say, or separate from perceptible objects (some say this too), or, if neither, then either they do not exist at all or they exist in some other way. So our debate will be not whether they exist, but in what way they exist' (*Metaphysics*, Book M, 1076a; the translation used here and subsequently is Annas 1976). One problem for Aristotle is that if we are to reject Platonic Forms, then what reason is there to believe in mathematical objects? What is their nature (if they exist), and, most important, what do we need mathematical objects for? What do they help to explain, or what do they shed light on? As he put it himself:

> One might also fix on this question about numbers: where are we to find reasons for believing that they exist? For someone who accepts Forms they provide some kind of explanation for things, since each number is a Form and a Form is an explanation of the being of other things somehow or other (we shall grant them this assumption). But what about the person who does not hold this sort of view through seeing the difficulties over Forms latent in it, so that this is not his reason for taking there to be numbers . . . ? Why should we credit him when he says that this sort of number exists, and what use is it to anything else? There is nothing which the man who believes in it says it causes . . . (*Metaphysics*, Book N, 1090a)

Aristotle's account of mathematical objects follows his account of Forms. As in the first quoted passage, he held that mathematical objects 'exist in perceptible objects', not separate from them. However, there is not much consensus over what this amounts to exactly. Some insight comes from a discussion in *Physics* B of what is distinctive about mathematical methodology:

The next point to consider is how the mathematician differs from the physicist. Obviously physical bodies contain surfaces, volumes, lines, and points, and these are the subject matter of mathematics . . . Now the mathematician, though he too treats of these things (viz., surfaces, volumes, lengths, and points), does not treat them as (*qua*) the limits of a physical body; nor does he consider the attributes indicated as the attributes of such bodies. This is why he separates them, for in thought they are separable from motion, and it makes no difference nor does any falsity result if they are separated . . . While geometry investigates physical lengths, but not as physical, optics investigates mathematical lengths, not as mathematical.(193b–194a)

Book M of the *Metaphysics* contains similar sentiments:

it is possible for there to be statements and proofs about perceptible magnitudes, but not as perceptible but as being of a certain kind . . . [I]n the case of moving things there will be statements and branches of knowledge about them, not as moving but merely as bodies, and again merely as planes and merely as lengths, as divisible and as indivisible but with position . . . [I]t is also true to say without qualification that mathematical objects exist and are as they are said to be . . . [T]he mathematical branches of knowledge will not be about perceptible objects just because their objects happen to be perceptible, . . . but neither will they be about other separate objects over and above these . . . So if one posits objects separated from what is incidental to them and studies them as such, one will not because of this speak falsely any more than if one draws a foot on the ground and calls it a foot long when it is not a foot long . . . A man is one and indivisible as a man, and the arithmetician posits him as one indivisible and studies what is incidental to man as indivisible; the geometer on the other hand studies him neither as a man nor as indivisible, but as a solid object . . . That is why the geometers speak correctly: they talk about existing things and they really do exist . . . (1077b–1078a)

Sticking to geometry for the moment, the idea here seems to be that physical objects somehow literally contain the surfaces, lines, and points studied in mathematics. The geometer, however, does not treat these surfaces, for example, as the surfaces *of* physical objects. In thought one can separate surfaces, lines, and points from the physical objects that contain them. This just means that we can focus on the surfaces, lines, and planes and ignore the fact that they are physical objects. This separation is psychological, or perhaps logical. It concerns how we think about physical objects. For Aristotle, Plato's mistake was to conclude that geometrical objects are metaphysically separate from their physical instantiations, just because mathematicians manage to ignore certain physical aspects of their subject-matter.

There are several interpretations of Aristotle here. One is to take the talk of mathematical *objects* seriously, and more or less literally. Accordingly, Aristotle postulated a faculty of *abstraction* whereby objects are created, or otherwise obtained or grasped, by contemplating physical objects. We abstract away some of their features (see, for example, Mueller 1970 and the Introduction to Annas 1976).

Suppose, for example, that we start with a brass sphere. If we selectively ignore the brass and focus only on the shape of the object, we will obtain the geometer's sphere. If we focus on the surface of one of the sides of an ice cube we get a segment of a plane and if we focus on an edge of this plane, we get a line segment. Thus, geometric objects are much like Forms. In a sense, geometric objects *are* the forms of physical objects. But, of course, they are Aristotelian and not Platonic Forms. The mathematical objects obtained by abstraction do not exist prior to, or independent of, the physical objects they are abstracted from.

On this interpretation, natural numbers are obtained via abstraction from collections of physical objects. We start with a group of, say, five sheep and selectively ignore the differences between the sheep, or even the fact that they are sheep. We focus only on the fact that they are different objects, and arrive at the *number* 5, which is a form, of sorts, of the group. So numbers exist, as Aristotelian Forms, in the groups of objects of which they are the numbers.

Notice that arithmetic and geometry come out literally true on a reading like this, pending an acceptable account of abstraction. Geometry is about geometrical objects, which have the properties ascribed to them in geometry treatises. Arithmetic is about natural numbers.[6] This is a pleasing realism in truth-value and a realism

[6] One unfortunate (if not damning) consequence of this account is that a natural number does not exist unless there is a collection of physical objects of that size. Similarly, a geometric object, such as a given polygon, exists only if there is a physical object that has that shape.

in ontology, consistent with passages like 'the geometers speak correctly: they talk about existing things and they really do exist . . .' (*Metaphysics*, M1078a).

Some interpreters have Aristotle distinguish the 'sciences' on the basis of their degree of abstraction from matter. Accordingly, physics concerns matter in motion, abstracting from the kind of matter it may be. Mathematics concerns matter as (geometric or numerical) quantity, abstracting from motion. Metaphysics is about being as such, abstracting from everything else.

This sort of abstraction has been roundly criticized throughout the history of philosophy. If I may be permitted a jump of about 2,000 years, one of the sharpest broadside attacks against abstraction was launched by the logician Gottlob Frege (writing about some of his contemporaries). Frege (1971: 125) discusses the so-called process whereby we take a group of 'counting blocks' and abstract away from the differences between them, so that the blocks become 'equal', much like Plato's ideal units. Supposedly, we then arrive at their number, as on the present reading of Aristotle. Frege replies that if, through abstraction, 'the counting blocks become identical, then we now have only one counting block; counting will not proceed beyond "one". Whoever cannot distinguish between the things he is supposed to count, cannot count them either.' That is, if we do manage to abstract away the differences between the blocks, then we cannot differentiate them, in order to count them:

> If abstraction caused all differences to disappear, it would do away with the possibility of counting. On the other hand, if the word 'equal' is not supposed to designate identity, then the objects that are the same will therefore differ with respect to some properties and will agree with respect to others. But to know this, we do not have to first abstract from their differences . . . [A]bstraction is nondistinguishing and nonseeing; it is not a power of insight or of clarity, but one of obscurantism and confusion.

Frege (1980a: 84–85) makes a similar point with more sarcasm:

> Inattention is a very strong lye; it must not be applied at too great a concentration, so that everything does not dissolve, and likewise not too dilute, so that it effects a sufficient change in the things. Thus it is a question of getting the right degree of dilution; this is difficult to manage,

> and I at any rate have never succeeded . . . [Abstraction] is particularly effective. We attend less to a property, and it disappears. By making one characteristic after another disappear, we get more and more abstract concepts . . . Suppose that there are a black cat and a white cat sitting side by side before us. We stop attending to their colour, and they become colourless, but are still sitting side by side. We stop attending to their posture, and they are no longer sitting (though they have not assumed another posture), but each one is still in its place. We stop attending to position; they cease to have place, but still remain different. In this way, perhaps, we obtain from each of them a general concept of Cat. By continued application of this procedure, we obtain from each object a more and more bloodless phantom. Finally we thus obtain from each a *something* wholly deprived of content; but the *something* obtained from one object is different from the *something* obtained from the other—though it is not easy to see how.

See also Frege 1884: §§13, 34. To paraphrase Berkeley, abstracted items seem to be the ghosts of departed objects.

A second interpretation of Aristotle's remarks on mathematics is to demur from ontological abstraction, and thereby reject the realism in ontology. We do not get to geometrical or arithmetic *objects* via any process. Strictly speaking, there are no such objects. The trick is to maintain realism in truth-value and, thereby, the objectivity of mathematics. Jonathon Lear (1982) interprets Aristotle's geometer as studying specific aspects of (some) ordinary physical objects, perhaps along lines like those suggested by Frege. Consider, once again, a sphere made of brass. The geometer does not abstract from the brass to arrive at a geometrical sphere. She simply ignores the brass and only considers properties of the physical object that follow from its being spherical. Whatever conclusion she draws will hold of a wooden sphere as well.

As indicated by the above passages, it is typical for a geometer to assume that there is a geometric object that has all and only the properties that we attribute to the sphere. This is to postulate special geometric objects, against this interpretation of Aristotle. However, Aristotle notes that the postulation of geometric objects is harmless, since the real physical sphere also has all of those properties we attribute to the postulated sphere. Strictly and literally, the geometer speaks only of physical objects (albeit not 'as physical'). However, it is harmless to pretend that the geometric sphere is separate. In other words, the objects of geometry are useful fictions. Suppose a geometer says, 'let *A* be an isosceles triangle'. He then attributes to *A* only properties that follow from its being an isosceles triangle. Mathematicians sometimes say that *A* is an 'arbitrary' isosceles triangle, but all they mean is that *A* could be any such triangle. By analogy with the present account, it would be a harmless fiction to say instead that *A* is a special object that has all the properties common to all isosceles triangles.

A similar account of arithmetic would come from treating a given object in a collection 'as indivisible' or 'as a unit'. In the collection of five sheep, for example, we regard each sheep as indivisible. Of course, as butchers know, each sheep is quite divisible, and so the mathematician's assumption is false. The idea is that the mathematician ignores any properties of the collection that arise from the divisibility of the individual sheep. We pretend that each sheep is indivisible, and so we treat it as indivisible.

Aristotle agrees with Plato that number is always a number *of* something, but for Aristotle numbers are numbers of collections of ordinary objects. Aristotle's numbers are Plato's physical numbers. As with geometry, it is harmless to introduce numbers as useful fictions, in giving the heuristics of arithmetic.

On both interpretations of Aristotle's philosophy of mathematics, the applicability of mathematics to the physical world is straightforward. The mathematician studies real properties of real physical objects. There is no need to postulate a link between the mathematical realm and the physical realm, since we do not deal with two separate realms. This is a seed of empiricism, or at least certain forms of it.

Unlike Plato, both interpretations of Aristotle make sense of the dynamic language that is typical of geometry. Since geometry deals with physical objects or direct abstractions from physical objects, talk of 'squaring and applying and adding and the like' is natural. We certainly do 'square and apply and add' physical objects and this talk carries over almost literally to geometry. Consider Euclid's principle that between any two points *one can draw* a straight line. For Plato, this is a disguised statement about the existence of Lines. Aristotle could treat the principle literally, as a statement of permissions indicating what one can *do*.

There is a potential problem concerning the *mismatch* between real physical objects and geometric objects or geometric properties.

This, of course, is an instance of the mismatch between object and Form that motivates Platonism. Consider the brass sphere and the side of the ice cube. The sphere is bound to contain imperfections and the surface of the cube is certainly not completely flat. Recall the theorem that a tangent to a circle intersects the circle in a single point (see Fig. 3.2 above). This theorem is false concerning real circles and real straight lines. So what are we to make of Aristotle's claim that 'mathematical objects exist and are as they are said to be', and the statement that 'the geometers speak correctly'?

On the abstractionist interpretation, we want to end up with objects that *exactly* meet the mathematical description of spheres, planes, and lines. To accomplish this, we have to abstract from any imperfections in the physical specimens, such as bumps on the surface of the cube. That is, we not only abstract from the brass, we abstract from the imperfections to arrive at a perfect sphere. If this further abstraction is allowed, then one might wonder how Aristotle's view differs from Plato's. In what sense are the final abstracted figures still part of the physical world? How do the perfect Forms exist *in* the imperfect physical objects? We seem to have re-entered Plato's world of Being, through the back door, or at least we encounter the major problems with the world of Being. The contemplated manoeuvre severs the intimate tie between mathematics and the physical world noted above.

On the second (fictionalist) interpretation, the geometer studies the consequences of a certain limited set of properties of physical objects. To solve the mismatch problem, Aristotle might hold that there are physical objects that lack the imperfections. In other words, there are physically real perfect spheres, cubes with perfectly flat surfaces and perfectly straight edges, perfect triangles, and so on. Aristotle did hold that heavenly bodies are (perfect) spheres and their orbits are spherical. However, the heavens do not give us enough objects for a rich geometry, and this suggestion does not account for the application of geometry here in the sub-lunar realm. It might be enough for Aristotle to hold that it is *possible* for there to be perfect spheres, lines, planes, and the like—even if there are none (or few) actual objects for the mathematician to study. Much of geometric demonstration proceeds via construction. The reader is asked to produce a certain straight line or circle. On the second interpretation, Aristotle must allow that this construction is possible—in the physical world using only physical tools. Similarly, in arithmetic the successor principle is affirmed when we note that for any possible collection of physical objects, there *could be* a collection with one more object. This move to modality could bring back the epistemic problems with Platonism. Aristotle might point out that geometry is applicable to the material world to the extent that the objects thereof *approximate* the perfect objects described in mathematical treatises, but this response is available to Plato as well.

One might think of the perfect objects of geometry (and arithmetic) as parts of physical space, but, as above, this would sever the tie with observed objects. Ideal circles and lines would not be 'in' the objects we see.

As noted, Aristotle shares with empiricism a close tie between the subject-matter of mathematics and the physical world. Such views founder on branches of mathematics that do not have such a direct connection to the material universe. Aristotle held that rational numbers are not numbers, but are related to natural numbers as ratios. Perhaps rational and even real analysis could emerge from an Aristotelian understanding of geometry. Following Euclid, one can either develop a theory of ratios of line segments or else recapture the real numbers via line segments, taking an arbitrary line segment as unit (along the lines of what Aristotle says about arithmetic units). However, at least prima facie, this is about as far as such a view can go. How would an Aristotelian understand complex analysis, or functional analysis, or point set topology, or axiomatic set theory? Of course, it is not fair to fault Aristotle for this lacuna, but any modern Aristotelians would have to face this problem.

5. Further Reading

Plato's remarks about mathematics are scattered throughout the dialogues, but mathematics comes in for special attention in the *Republic* and *Theaetetus*. Aristotle's philosophy of mathematics is found mostly in *Metaphysics* M and N, especially chapter 3 of M. Annas 1976 is a readable translation, and it contains a lucid account of Plato's and Aristotle's philosophy of mathematics. A standard source for Plato on mathematics is Wedberg 1955; see also Vlastos

1991: ch. 4, Mueller 1992, and Turnbull 1998. A standard source for Aristotle on mathematics is Apostle 1952; see also Lear 1982 and Mueller 1970.