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LOGICISM: IS MATHEMATICS (JUST) LOGIC?

> Mathematics and logic, historically speaking, have been entirely distinct studies . . . But both have developed in modern times: logic has become more mathematical and mathematics has become more logical. The consequence is that it has now become wholly impossible to draw a line between the two; in fact the two are one . . . The proof of their identity is, of course, a matter of detail.
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> (Russell 1919: ch. 18)

THE previous chapter presented Immanuel Kant's views that (1) mathematics is knowable independent of sensory experience—mathematics is a priori—and (2) the truths of mathematics cannot be determined by analysing concepts—they are synthetic. Although one can hardly overestimate Kant's influence, subsequent philosophers had difficulty squaring these views with developments in mathematics and science. As noted above, Alberto Coffa (1991) argued that a main concern of nineteenth-century philosophy was to account for the prima facie necessity and a priori nature of mathematics and logic without invoking Kantian intuition, or some other reference to a priori forms of spatial and temporal intuition. The two alternatives to Kant's view seem to be that mathematics is empirical (and so a posteriori) and that mathematics is analytic. Section 3 of the previous chapter sketched John Stuart Mill's bold attempt at the former alternative. We now move forward a few decades, to near the turn of the twentieth century, and consider views that mathematics is analytic (or all but analytic). Some of the authors examined in this chapter hold that at least parts of mathematics are, or can be reduced to, logic. The idea is that the concepts and objects of mathematics, such as 'number', can be defined from logical terminology; and with these definitions, the theorems of mathematics can be derived from principles of logic. The view is called 'logicism'. We begin with Gottlob Frege, the first accomplished mathematician we meet in our historical survey (other than the passing mention of the rationalists Descartes and Leibniz).

1. Frege

We must briefly attend to the changing notions of analyticity and a priori knowledge. These mean different things to different thinkers. Recall that for Kant, if a proposition is in subject-predicate form, then it is analytic if its subject concept contains its predicate concept.[1] The central idea is that analyticity turns on the metaphysics of *concepts*. One determines whether a proposition is analytic by analysing its concepts.

Frege employed a different, but perhaps related distinction. He held that analyticity is like a priority in being an *epistemic* concept, turning on how a given proposition is known (or knowable):

> [T]hese distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement. Where there is no justification, the possibility of drawing the distinctions vanishes. When . . . a proposition is called a posteriori or analytic in my sense, . . . it is a judgement about the ultimate ground upon which rests the justification for holding it to be true . . . The problem becomes . . . that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one . . . If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some general science, then the prop-

[1] One of Frege's innovations was to dislodge philosophers from the dominance of the subject-predicate form of propositions. Instead, he thought of each proposition as decomposable into function and argument in a variety of ways, a notion he borrowed from mathematics.

osition is a synthetic one. For a truth to be a posteriori, it must be impossible to construct a proof of it without including an appeal to facts, i.e., to truths which cannot be proved and are not general . . . But if, on the contrary, its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof, then the truth is a priori. (Frege 1884: §3)

Although Frege believed that every knowable proposition has an 'ultimate ground', something like a canonical proof, the crucial philosophical definitions can be formulated without presupposing this. A proposition is a priori if either it is an unprovable 'general law' or it has a justification—proof—which relies only on such unprovable general laws. A proposition is analytic if either it is a 'general logical law or definition' or it has a proof that relies only on such general logical laws and definitions.[2] There is a particularly logical source of knowledge, and the analytic truths are known on that basis.

The above passage indicates that Frege held that only knowable or justifiable propositions can be analytic or a priori. Since he also held that arithmetic and real analysis are analytic, he believed that every truth about the natural numbers and every truth about the real numbers is knowable. That is, every such truth is either provable or an unprovable general logical law or definition. Frege was committed to the view that for every proposition about the natural numbers or the real numbers, either it or its negation is knowable.

To show that arithmetic propositions are analytic, Frege had to show how to derive them from general logical laws and definitions. His logicist programme was an attempt to do just that, at least for the basic principles of the field.

Frege began with a general fact about counting. Someone can determine if two collections are the same by putting them in one-to-one correspondence. Let us say that two concepts are *equinumerous* if there is a one-to-one correspondence between the objects falling under one and the objects falling under the other. For example, on a set table the napkins are equinumerous with the plates if there is exactly one napkin corresponding to each plate. In a monogamous society the husbands are equinumerous with the

[2] This raises a question about the 'general (logical) laws and definitions'. How are those known? To what extent are they a priori? Perhaps Frege took general laws and definitions to be self-evident, or self-evidently a priori.

wives (by definition). Despite the particle 'numerosity' in the name, Frege showed how to define equinumerosity using only the resources of (so-called 'higher-order') logic, without presupposing natural numbers, or the notion of number generally. He (1884: §63) proposed the following thesis, now known as 'Hume's principle':[3]

For any concepts *F*, *G*, the number of *F* is identical to the number of *G* if and only if *F* and *G* are equinumerous.

As Frege intends it, the phrase 'the number of *F*' is a grammatical form for denoting an *object*. That is, 'the number of *F*' is a proper name (broadly speaking), or what is today called a 'singular term'. In the terminology of Chapter 2, §2.1 above, Frege was a realist in ontology, believing in the independent existence of the natural numbers. He was also a realist in truth value, holding that statements of mathematics have objective truth values.

Let *Z* be the concept 'not identical to itself'. Since every object is identical with itself, no object has the concept *Z*. That is, for every object *a*, *Za* is false. Frege defined the number zero to be the number of the concept *Z*.

Frege (Frege 1884: §76) then defined the successor relation among numbers. The 'number *n* follows in the series of natural numbers directly after *m*' if and only if

there exists a concept *F*, and an object falling under it *x*, such that the number which belongs to the concept *F* is *n* and the number which belongs to the concept 'falling under *F* but not identical to *x*' is *m*.

In other words, *n* is a successor to *m* if there is a concept which applies to exactly *n* objects and when we remove one of those objects, *m* objects remain. Frege's precise language is designed to say this using only logical terminology like 'object', 'concept', and 'identity'.

Let *T* be the concept 'identical with zero', so that for any object *b*, *Tb* holds if and only if $b = 0$. That is, *T* holds of exactly one thing, the number zero. Frege defined the number one to be the number

[3] The name follows Frege's citation of a similar principle by the eighteenth-century empiricist David Hume. Fregean *concepts* exist objectively, and so are not mental entities, but they can be grasped through the mind. In the terminology of contemporary philosophy, 'property' might be a better term than 'concept' here.

of the concept T. He showed that the number one 'follows zero in the series of natural numbers', according to his own definition.

Frege reminded the reader that this 'definition of the number 1 does not presuppose, for its objective legitimacy, any matter of observed fact'. In other words, the underlying propositions are a priori and objective.

The next step is to define the number two to be the number of the concept 'either identical to zero or identical to one', and so on for the rest of the natural numbers. In general, let n be any number in the series of natural numbers. Consider the concept S_n, 'member in the series of natural numbers ending with n'. That is, for any object a, $S_n a$ holds if and only if a is a natural number less than or equal to n. Frege showed that the number of the concept S_n is a successor to n: the number of S_n is $n + 1$. This establishes that there are infinitely many natural numbers.

It remains to give a definition of *natural number*. One would like to say that n is a natural number if n is obtained from the number zero after finitely many applications of the successor operation. As a definition, however, this would be circular, since it invokes the notion of 'finitely many'. Frege devised a way to accomplish the definition using only logical resources. To paraphrase, n is a natural number if and only if

> For any concept F, if F holds of the number zero and if for every object d, from the proposition that d falls under F it follows that every successor of d falls under F, then n falls under F.

In more contemporary terms, n is a natural number if n falls under every concept which holds of zero and is closed under the successor relation. In symbols:

$$Nn \equiv \forall F[(F0 \,\&\, \forall d \forall d'(\,(Fd \,\&\, \text{'}d'\text{' is a successor of } d\text{'}) \rightarrow Fd')\,) \rightarrow Fn].$$

Frege then showed how common arithmetic propositions, such as the induction principle, follow from these definitions. The derivation of the basic principles of arithmetic from Hume's principle is now known as *Frege's theorem*.

Frege was not satisfied with this development. Hume's principle determines identities of the form 'the number of F = the number of G', where F and G are any concepts, but it does not determine the truth value of sentences in the form 'the number of F = t', where t is an arbitrary singular term. In particular, Hume's principle does not determine whether the number 2 is identical with a given set, or with Julius Caesar. I presume that no one is going to confuse the number 2 with the emperor, but Hume's principle itself does not settle the question.

To sum up, so far Frege has (brilliantly) determined the relations between the natural numbers, and provided adequate definitions of the sizes of various collections, all from Hume's principle, but he has not *identified* the natural numbers. What, after all, *is* the number 2? The underlying idea is that we have not succeeded in characterizing the natural numbers *as objects* unless and until we can determine how and why any given natural number is the same or different from any object whatsoever. To borrow a slogan from W. V. O. Quine, 'no entity without identity'. In the context of Fregean logicism, the problem of identifying the natural numbers has become known as the 'Caesar problem' (see Heck 1997a).

Notice that the development so far takes Frege's principle as an unjustified starting-point. It is part of Frege's methodology that one should try to prove what one can, and thus reveal its epistemic ground. He attempted to do so for Hume's principle.

The *extension* of a concept is the class of all objects that the concept applies to. For example, the extension of 'chair' is the class of all chairs. Frege (Frege 1884: §68) defined natural numbers in terms of concepts and their extensions:

The number which belongs to the concept F is the extension of the concept equinumerous with the concept F'.

The number two, for example, is the extension (or collection) containing all concepts that hold of exactly two objects.[4] So the concept of being a parent of Aviva Shapiro is a member of the number two, as is the concept of being a shoe on a given fully-dressed person, and the concept of being a prime number less than five. The number three is the extension (or collection) containing all concepts that hold of exactly three objects, and so on.

Frege (1884: §73) showed how Hume's principle follows from these definitions and some common properties of extensions. With

[4] It is interesting that Frege did not raise a Caesar-type problem for extensions. How do we know, for example, whether Caesar is the extension of those concepts that hold of exactly two objects? Since extensions are closely linked to concepts, perhaps Frege took them to be already known.

Frege's theorem, this completes the derivation of arithmetic, and the establishment of logicism for the natural numbers—provided that the above definitions are correct. Under these assumptions, Frege succeeded in showing that arithmetic is analytic. The account proceeded through a rigorous and eminently plausible account of the application of arithmetic to the counting of concepts and collections of objects.

One cannot overestimate Frege's accomplishment. Who would have thought that so much could be derived from so little and, in particular, from such simple and obvious facts about concepts, extensions, and counting? However, arithmetic is only an early part of mathematics. Frege's plans to extend logicism to real analysis were not developed into a detailed programme (see, for example, Simons 1987 and Dummett 1991: ch. 22). One can only speculate on the extent to which Fregean logicism might accommodate some of the contemporary branches of mathematics, such as complex analysis, topology, and set theory.

A reader familiar with contemporary logic might notice an incongruity in Frege's logicism. The thesis that principles of arithmetic are derivable from the laws of logic runs against a now common view that logic itself has no ontology. There are no particularly logical objects.[5] From this perspective, logicism is a non-starter, at least for an ontological realist like Frege, who held that natural numbers exist as independent objects. There are infinitely many natural numbers, and so if logic says nothing about how many objects there are, then one cannot define the natural numbers in logic.

Frege, however, followed a tradition that concepts are in the purview of logic, and, for Frege, extensions are tied to concepts. So logic does have an ontology. Logical objects include the extensions of some concepts that exist of necessity. Thus, logical objects exist of necessity, and so the necessity of logic is maintained.

As indicated from the first quoted passage above, Frege explicitly distinguished logic from special sciences, such as physics. Logic is topic-neutral since it is universally applicable; logical truths are

[5] As we saw in §2 of the previous chapter, the pedigree for this view traces to Kant. In discussing a particular argument for the existence of God, Kant claimed that analysis of concepts cannot entail the existence of anything. If Kant is correct about this, and if logic consists of conceptual analysis, then there are no specifically logical objects.

absolutely general. The use of concepts—and their extensions—does not undermine this neutrality. One needs to deal with concepts in order to think at all. For any sort of objects, there are concepts of those objects and extensions of those concepts. Frege showed how to construct the natural numbers from this logical ontology. He also pointed out that arithmetic enjoys the universal applicability of logic. Any subject-matter has an ontology, and if one has objects at all, one can count them and apply arithmetic.

We should note that Frege did not extend his logicism to geometry. On that score he was a Kantian, holding that the principles of Euclidean geometry are synthetic a priori (with those notions understood in a Fregean sense, as above). Frege held that geometry does have a special, non-universal subject-matter—space. We need not further pursue these issues concerning the boundaries of logic (see Shapiro 1991: chs. 1–2). There are larger issues on the horizon.

Even limited to arithmetic—and waiving the boundary issues—it is sad to report that our story does not have a tidy and compelling ending. Frege's later *Grundgesetze der Arithmetik* (1893, 1903) contains a full development of a theory of concepts and their extensions. For present purposes, the crucial plank is the now infamous Basic Law V, paraphrased as follows:

> For any concepts *F*, *G*, the extension of *F* is identical to the extension of *G* if and only if for every object *a*, *Fa* if and only if *Ga*.

In other words, the extension of *F* is the extension of G if and only if *F* and *G* hold of the same objects.

A letter from Bertrand Russell in 1902 (see van Heijenoort 1967: 124–5) revealed that Basic Law V is inconsistent.[6] Let *R* be the concept that applies to an object *x* just in case

> there is a concept *F* such that *x* is the extension of *F* and *Fx* is false.

Let *r* be the extension of *R*. Suppose that *Rr* is true. Then there is a concept *F* such that *r* is the extension of *F* and *Fr* is false. It follows from Basic Law V that *Rr* is also false (since *r* is also the extension of *R*). Thus if *Rr* is true, then *Rr* is false. So *Rr* is false. Then there is a concept *F* (namely *R*) such that *r* is the extension of *F* and *Fr* is false. So, by definition, *R* holds of *r*, and so *Rr* is true. This is a contradic-

[6] The mathematician Ernst Zermelo discovered the paradox about a year earlier. See Rang and Thomas 1981.

tion, and so Basic Law V is inconsistent. This is now known as *Russell's paradox*.

Frege took this paradox to be devastating to his logicist programme. Nevertheless, he sent Russell a gracious reply, almost immediately:

> Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic . . . [The matter is] all the more serious since, with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish . . . In any case your discovery is very remarkable and will perhaps result in a great advance in logic, unwelcome as it may seem at first glance. (van Heijenoort 1967: 127–8)

In the same letter, Frege gave a more accurate formulation of the paradox. After some attempts to recover from the blow, Frege abandoned his logicist project, left in ruins. We turn to others who took up the mantle of logicism, starting with Russell himself.

2. Russell

Russell (1919: ch. 2) held that Frege's account of the natural numbers is substantially correct:[7]

> The question 'What is number?' is one which has been often asked, but has only been correctly answered in our own time. The answer was given by Frege in 1884, in his *Grundlagen der Arithmetik*. Although this book is quite short, not difficult, and of the very highest importance, it attracted almost no attention, and the definition of number which it contains remained practically unknown until it was rediscovered by the present author in 1901.

Russell added a footnote that the same definitions are 'given more fully and with more development' in Frege (1893) and (1903). We may conclude that Russell did not accept Frege's assessment that

[7] In discussing Frege's seminal logical work, *Begriffsschrift* (1879), Russell (1919: ch. 3) said that in 'spite of the great value of this work, I was, I believe, the only person who ever read it—more than twenty years after its publication'.

'the sole possible foundations of arithmetic seem to vanish' in the contradiction from Basic Law V.

In fact, Russell held that once it is properly understood, Basic Law V is correct as a definition of 'extension' or 'class'. His diagnosis was that the derivation of the contradiction from Basic Law V invokes a fallacy. Recall (from ch. 1, §2) that a definition of a mathematical entity is *impredicative* if it refers to a collection that contains the defined entity. The usual definition of the 'least upper bound' is impredicative since it refers to a set of upper bounds and characterizes a member of this set.

Russell (1919: ch. 17) argued that such definitions are illegitimate, since they are circular:

> Whenever, by statements about 'all' or 'some' of the values that a variable can significantly take, we generate a new object, this new object must not be among the values which our previous variable could take, since, if it were, the totality of values over which the variable could range would be definable only in terms of itself, and we should be involved in a vicious circle. For example, if I say 'Napoleon had all the qualities that make a great general', I must define 'qualities' in such a way that it will not include what I am now saying, i.e., 'having all the qualities that make a great general' must not be itself a quality in the sense supposed.

The development of Russell's paradox runs foul of the 'vicious circle principle'. To generate the paradox, we defined a concept R which 'applies to an object x just in case there is a concept F such that x is the extension of F and Fx is false'. The definition of R refers to all concepts F, and R is just such a concept F. Thus, the definition of R is impredicative. We derive a contradiction from the assumption that the definition of R holds of its own extension. The ban on impredicative definitions precludes even making this assumption.

For now, let us put concepts aside and speak only of extensions, or classes. Russell argues, from the vicious circle principle, that it 'must under all circumstances be meaningless (not false) to suppose [that] a class [is] a member of itself or not a member of itself'. Thus, there can be no all-inclusive class that includes all of the classes in the universe, since this domain would be a member of itself. Nor can there be a class of all classes that do not contain themselves as members. For Russell, it is *meaningless* to say (or even assume) that there is such a class. He proposed a *type theory*, which partitions the universe. Define an 'individual' to be an object that is not a class. Individuals are of type 0, and classes of individuals are of type 1. Classes of classes of individuals are of type 2, and so on. So, for example, the people that make up a baseball team are each individuals and so are type 0 objects. The team, regarded as a class of its players, is a type 1 object; and the league, regarded as a class of teams, is of type 2. A collection of leagues would be of type 3.

The move to classes allows a simplification of Frege's definitions of the natural numbers. For any class *C*, define the *number of C* to be the 'class of all those classes that are' equinumerous with *C* (see Russell 1919: ch. 2). Let *A* be the class of my three children; so that *A* is of type 1. The number of *A* is the class of all three-membered type 1 classes. The number of my children is thus a type 2 class. Similarly, the number of a type 2 class is a type 3 class, and so on. For Russell, a 'number is anything which is the number of some class'. He defined the number zero to be the class of all type 1 classes that have no members. So zero is a type 2 class which has exactly one member—the type 1 empty set. The number 1 is the class of all type 1 classes that have a single member. The number 1 is also a type 2 object, and it has as many members as there are individuals (if this statement mixing types is allowed).[8] Continuing, the number 2 is the class of all type 1 classes that have two members. Thus, the number 2 is the class of all pairs of individuals. The number 3 is the class of all triples of individuals, and so on. As expected, the number of the aforementioned class *A* of my children is 3.

Russell adapted another central Fregean definition to the context with classes: 'the *successor* of the number of . . . [a] class α is the number of . . . the class consisting of α together with x, where x is [any individual] not belonging to [α]' (1919: ch. 3). So far, so good.

Recall that, for Frege, the number zero is the number of the concept 'not identical to itself'. This conforms to Russell's programme in which zero is a type 2 class. However, Frege's presentation of the other natural numbers, and his proof (via Hume's principle) that there are infinitely many natural numbers, violates Russell's type restrictions (and the vicious circle principle). Recall that Frege proposed that the number 1 is the number of the concept

[8] There are different natural numbers for each type. We might define 0^1 to be the class of all type 2 classes that have no members, and 1^1 to be the class of all type 2 classes with a single member, etc. So 0^1 and 1^1 are of type 3.

'identical with zero'. Using classes instead of concepts, the number 1 would be the number of the class whose only member is the number zero. That is, Frege's number 1 is the number of {0}. But {0} is of *type 3* and so the number of this class is of *type 4*. Notice that even though the number 0 has a single member (i.e., the type 1 empty set), 0 is not a member of Russell's number 1, since the latter contains only type 1 classes—as per the type restrictions. Since the number 0 is of type 2, it is a member of the *type 3* class consisting of all type 2 classes that have a single member (see note 8).

To help keep the types straight, let us temporarily define 1_R, the Russell-1, to be the type 2 class consisting of all type 1 classes with a single member; and define 1^1 to be the *type 3* class consisting of all type 2 classes that have a single member. So Russell's number zero is a member of 1^1 but not a member of 1_R. For Frege, the number two is the number of the concept 'either identical to zero or identical to one'. Transposing this to the present context (involving classes instead of concepts), Frege's number two would be the number of the class {0,1}. Which number 1, 1_R or 1^1? It does not work either way. For Russell, the class $\{0,1^1\}$ does not exist, since it contains a type 2 class and a type 3 class.[9] The class $\{0,1_R\}$ contains a pair of type 2 classes and so it is of type 3. The number of $\{0,1_R\}$ is thus of type 4. In general, Frege's plan to define a number *n* as the number of the predecessors of *n*: $\{0,1, \ldots n - 1\}$ runs into trouble. We either violate the type restrictions directly (if 0, 1, etc. are not all of the same type) or else we produce a class of the wrong type.

For Russell, again, each number *n* is the *type 2* class consisting of all *n*-membered classes of (type 0) individuals—that is, all *n*-membered classes of non-classes. He could not accept Frege's proof that there are infinitely many natural numbers, for that involved treating the natural numbers as if they were individuals. Like Basic Law V, Frege took Hume's principle to be impredicative.

[9] With some care, it is possible to consistently define classes of mixed type, such as a class of players and teams. Contemporary Zermelo–Fraenkel set theory allows mixed classes and so it has a class of all classes of finite type, and then subclasses of that, etc. The resulting structure is sometimes called the 'cumulative hierarchy'. Allowing mixed types facilitates the extension of the type hierarchy beyond finite types. In the cumulative hierarchy, there is no set of all sets that are not members of themselves. There is no universal set, containing all sets as members, and there is no set of all singletons. So the Fregean construction is blocked there too.

Frege's theorem, including the poof that there are infinitely many natural numbers, turns on this impredicativity.

For Russell, whether a given natural number exists depends on how many *individuals* (i.e. non-classes) there are in the universe. Suppose, for example, that the world contains exactly 612 individuals. Then Russell's number 612 would be the class of all 612-membered classes of individuals. There would be only one such class, the class of *all* individuals. To follow the definition, the successor of 612 is the number of 'the class consisting of' the universe 'together with *x*, where *x* is [any individual] not belonging to' the universe. Well, under the assumption about the size of the universe, there is no such *x* and so there is no successor of 612. The numbers simply run out at 612—there is no number 613.

To avoid this embarrassment, Russell and Whitehead propose an *axiom of infinity*, which states that there are infinitely many *individuals*. Russell admits that this principle does not enjoy the epistemic status of the other fundamental principles he employs (such as the definitions). The axiom of infinity cannot be proved, nor is it analytic, a priori, true of necessity. Nevertheless, it seems to be essential *for arithmetic*, so Russell accepts it as a postulate. The existence of each natural number, and its successor, then follows.

The contrast with Frege is stark. Frege *proved* that each natural number exists, but his proof is impredicative, violating the type restrictions. Russell had to *assume* the existence of enough individuals for each natural number to exist. This puts a damper on logicism. If we go on to prove an arithmetic theorem Φ, all we can say is that the statement

if there are infinitely many (type 0) individuals, then Φ

is a theorem of logic. Most of arithmetic has an awkward, hypothetical status.

With the axiom of infinity on board, the next step is to define the general notion of *natural number*. Here again, Russell attempts to transpose Frege's proposal to the context of classes: *n* is a natural number if *n* belongs to every (type 3) class which contains the number 0 and also contains a successor of each of its members. As it stands, however, this definition is impredicative, in a most straightforward manner. The class of natural numbers is a type 3 class defined by referring to 'every class' of that type. To maintain full compliance with the vicious circle principle, Russell (and

Whitehead) insisted on further structure in the hierarchy of types. A type 1 class is 'predicative', or of level 0, if it can be defined without referring to classes. A type 1 class is of level 1 if it is not predicative, but can be defined with reference to predicative classes only. A type 1 class is of level 2 if it is not of level 1 but can be defined with reference to level 1 classes only. There is a similar level structure for every type. The overall theory is sometimes called 'ramified type theory'.[10]

In the foregoing definition of 'natural number', the locution 'every class' would have to be restricted to a certain *level* in the ramified hierarchy of type 2 classes. One should say that *n* is a type 2, level 1 natural number if *n* belongs to every predicative class which contains the number 0 and also contains a successor of each of its members; *n* is a type 2, level 2 natural number if *n* belongs to every level 1 class which contains the number 0 and also contains a successor of each of its members; and so on. However, we now have no reason to think we get the same class of 'natural numbers' at each level. Russell and Whitehead realized that they could not sufficiently develop mathematics with the level restrictions, since some of the crucial definitions seem to require impredicative definitions. For example, Frege's proof of the induction principle for natural numbers from these definitions does not go through. When formulated in Russell's system, the induction principle is, or appears to be, impredicative, and many important mathematical developments are impredicative.

In response to this difficulty, Russell and Whitehead proposed another axiom, a principle of *reducibility* which states that at each type, for every class *c*, there is a predicative (level 0) class *c′* which has the same members as *c*. The principle of reducibility states that no new classes are generated beyond the first level. This allowed Russell and Whitehead to restrict the locution 'all classes' to 'all predicative classes', and then proceed with the derivation of the basic principles of arithmetic. The effect of the principle of reducibility is to allow the logician to ignore the level-hierarchy and proceed as if impredicative definitions are acceptable and the vicious circle is not really a problem. A nice deal, if you can get it.

[10] Whitehead and Russell 1910. See Hazen 1983 for a readable and sympathetic development of ramified type theory. Russell used the word 'order' for what I call 'level' here. In the contemporary literature, a phrase like 'second-order' or 'higher-order' refers to something more like a type in Russell's hierarchy.

But what is the status of the principle of reducibility? Is it analytic? Knowable a priori? Is it even true? Critics charged that the principle is ad hoc. Russell's response was the same as for the axiom of infinity. He admitted that the reducibility principle does not enjoy the same justification as the principles of logic, and he did not provide a compelling argument for it. Yet he claimed that it is essential for the development of mathematics, and so he proposed it as a postulate. He admitted that the axiom of reducibility is a flaw in his logicism.[11]

Using the principles of infinity and reducibility, Russell and Whitehead established the standard Peano axioms for arithmetic, and thus all of the usual theorems concerning the natural numbers. They then extended the development to some more advanced branches of mathematics, invoking ever higher types along the way. Let m be a natural number. Russell (1919: ch. 7) defined the *integer* $+m$ to be the binary 'relation of $n+m$ to n (for any n)' on the natural numbers. Thus, for example, +4 is the relation that holds of the following pairs: $\langle 4,0\rangle$, $\langle 5,1\rangle$, $\langle 6,2\rangle$, . . . Similarly, the integer $-m$ is the converse of $+m$, 'the relation of n to $n+m$', so that -4 holds of $\langle 0,4\rangle$, $\langle 1,5\rangle$, $\langle 2,6\rangle$, ... One can then define addition and multiplication on these 'integers' so that the usual properties hold.

It is widely thought, and widely taught, that the integers are an extension of the natural numbers. We go from the natural numbers to the integers by tacking on the negative whole numbers, so that the natural number 2, for example, is identical to the integer +2. Russell emphasized that on his definitions, the natural numbers and the integers are distinct from each other. The natural number 2 is a *class of classes* (i.e. a type 2 class) while the integer +2 is a *relation* on natural numbers. It would violate the type restrictions to identify this natural number with this integer: ' . . . $+m$ is under no circumstances capable of being identified with m, which is not a relation, but a class of classes. Indeed, $+m$ is every bit as distinct from m as $-m$ is'.

Next, the rational numbers are defined to be relations which

[11] F. P. Ramsey (1925) proposed a 'simple' or impredicative type theory without the restrictions on levels, but then presumably one needs to justify the violations to the vicious circle principle. Ramsey adopted an ontological realism towards classes, which obviates the need for a vicious circle principle. See ch. 1, §2 above. We return to this briefly later in this chapter.

capture *ratios* among integers: 'We shall define the fraction m/n as being that relation which holds between two [numbers] x,y when $xn = ym$'. Thus, for example, the fraction 3/4 is the relation that holds of the pairs: $\langle 3,4 \rangle$, $\langle 6,8 \rangle$, . . . Intuitively, the relation 3/4 holds between x and y just in case the fraction x/y reduces to 3/4. Notice also that the rational number $m/1$ is not the same relation as the integer $+m$. So the rational number 2 is different from the integer 2 and the natural number 2. One can define the 'greater than' relation and the operations of addition and multiplication, on these rational numbers, to recapture the arithmetic of the rational numbers.

For the real numbers, Russell follows another logicist, Richard Dedekind (1872). Define a 'section' to be a non-empty class c of rationals such that (1) for all rational numbers x,y, if x is in c and if $y < x$ then y is in c; (2) there is a rational number z such that for every rational number x if x is in c, then $x < z$; and (3) for every rational number x if x is in c then there is a rational number y in c such that $x < y$. In other words, a section is a connected, bounded class of rational numbers that has no largest member. The sections correspond to what are called 'Dedekind cuts' in the rational numbers. Russell identified the real numbers with the sections. The real number 2 is the class of rationals less than 2 (i.e. 2/1), and the square root of 2 is the class of all negative rational numbers together with the non-negative rationals whose square is less than 2. One can define the order relation on the real numbers, and the addition and multiplication operations, and then show that the real numbers are a complete ordered field. In particular, one can establish the completeness principle that every bounded class of real numbers has a least upper bound.

Notice that on this definition, real numbers are *classes* of rational numbers. The axiom of reducibility—or the use of impredicative definitions—thus plays a large role in the Russell's development of real analysis. It becomes impossible to keep the levels straight. For example, it will not do to have a level 0 square root of 2, a level 1 square root of 2, and so on. For real analysis, Russell also needed an axiom of choice, stating that for any collection c of non-empty classes, no two of which share a member, there is at least one class containing exactly one member of each member of c (Russell 1919: ch. 12; see Moore 1982 for a full development of the role of choice principles in the development of mathematics). Like infinity and reducibility, this axiom can be formulated using logical terminology, but perhaps not established from logical principles alone.

Finally, Russell defined a complex number to be an ordered pair of real numbers. So the complex number $3 - 2i$ is identified with the ordered pair whose first member is the real number 3 and whose second member is the real number -2.

This more or less completes the development of Russell's logicism. Russell (1919: ch. 18) asks a partly rhetorical question: 'What is this subject, which may be called indifferently either mathematics or logic? . . . Certain characteristics of the subject are clear. To begin with, we do not, in this subject, deal with particular things or particular properties: we deal formally with what can be said about *any* thing or *any* property.' Logic is completely general, and universally applicable.

To the extent that geometry concerns physical *space*, it falls outside the scope of Russell's logicism. However, one can consider a 'pure' version of geometry, which consists of pursuing the consequences of various axiom systems. This much can be fitted into the logicist framework, with the advent of model theory and the rigorous notion of logical consequence. With the principles of infinity, reducibility, and choice, Whitehead and Russell's type theory captures just about every branch of pure mathematics short of set theory.

But what is mathematics *about*? What are numbers, function, and so on really? Since Russell took the various sorts of numbers to be classes, relations on classes, relations on relations on classes, and so on, the status of numbers turns on the status of classes. His mature writings deny the independent existence of classes. In *Introduction to Mathematical Philosophy* (1919: ch. 18) he wrote that 'the symbols for classes are mere conveniences, not representing objects called "classes" . . . [C]lasses are in fact . . . logical fictions . . . [They] cannot be regarded as part of the ultimate furniture of the world'. Russell indicated (or tried to indicate) how to paraphrase any statement about classes as a statement about concepts and properties (what he called 'propositional functions'). The end result is what he called the 'no class' theory. Talk of classes is only a 'manner of speaking', and is eliminable in practice.[12]

[12] Here the axiom of reducibility plays an even larger role, since a single paraphrase might require one to speak at once of all concepts in the entire type hierarchy. Russell sometimes speaks of a 'systematic ambiguity', where the same sentence is used to express different propositions about each type and/or level.

Since Russell's numbers are classes (or constructed from classes), they are also logical fictions, and so not part of the 'ultimate furniture of the world'. So at this period, Russell sharply departed from Frege's realism in ontology. During his mature 'no class' period, he held that any statement in any branch of (pure) mathematics could be properly rewritten as a complex statement about properties and concepts, with no reference to numbers, functions, points, classes, and so on.

3. Carnap and Logical Positivism

We now consider a school of empiricism that flourished in the early and middle decades of the twentieth century. *Logical positivism* took off from the spectacular success of the natural sciences and the growth of mathematical logic. As noted earlier, mathematics is a difficult case for empiricism. In the previous chapter we considered Mill's view that the truths of mathematics are themselves known empirically, by generalizations on experience. Accordingly, mathematics is synthetic and a posteriori. In contrast, the logical positivists were attracted to the logicist thesis that the truths of mathematics are analytic, and so a priori. As we have seen already, these notions mean different things to different authors. We encounter a further evolution of the notion of analyticity.

As noted at the outset of this chapter, Coffa (1991) suggested that much of nineteenth-century philosophy was occupied with attempts to account for the (at least apparent) necessity and a priori nature of mathematics and logic without invoking Kantian intuition. Coffa suggested that the most fruitful anti-Kantian line was what he calls the 'semantic tradition', running through the work of Bernard Bolzano, the early Ludwig Wittgenstein, Frege, and David Hilbert, culminating with Moritz Schlick and Rudolf Carnap in the Vienna Circle. These philosophers developed and honed many of the tools and concepts still in use today, both in mathematical logic and in western philosophy generally. The main insight was to locate the source of necessity and a priori knowledge in the use of *language*. Necessary truth is truth by definition; a priori knowledge is knowledge of language use. Michael Dummett calls the approach the *linguistic turn* in philosophy.[13]

In the present context, the thesis is that once we understand the *meanings* of terms like 'natural number', 'successor function', 'addition', and 'multiplication', we would thereby have the resources to see that the basic principles of arithmetic, such as the induction principle, are true. This is at least in the spirit of logicism, even if, strictly speaking, the truths of mathematics do not end up being true on logical grounds alone.

Of the two main logicists considered above, Frege held that the numbers exist, of necessity, independent of the mathematician, and Russell held that numbers do not exist (at least during his no-class period). One might think that this exhausts the options, but as an empiricist Carnap found the whole metaphysical question of the existence of numbers troubling. How can that issue be decided by observation? Carnap rejected the sense of the very debate over the existence of mathematical objects.

On one level, the ontological question has a trivial, affirmative answer. 'There are numbers' is a logical consequence of 'there are prime numbers greater than 10'. If we accept the latter, as surely we must if we take mathematics and science seriously, then we accept the former: a tidy end to a 2,000-year-old struggle. Frege and Plato win; Russell, Mill, and perhaps Aristotle lose.

Of course, the ontological anti-realist would not be moved by this simple logical inference, and many ontological realists agree that the issue is not that simple.[14] So what is the traditional dispute about? Carnap (1950: §2) suggested that the parties 'might try to explain what they mean by saying that it is a question of the ontological status of numbers; the question whether or not numbers have a certain metaphysical characteristic called reality . . . or subsistence or status of "independent entities"'. Carnap complained

[13] Dummett locates the 'linguistic turn' with Frege, but this is controversial. Although Frege was clearly a pivotal figure in the eventual development of the semantic tradition, he did not hold that all necessary truth is truth by definition. Recall that, for Frege, the truths of geometry are *synthetic*, a priori (see §1 above), and so not true by definition. For Frege, analytic truths are derivable from 'general logical laws and definitions'. Thus, the status of Fregean analytic truths turns on the nature of 'general logical laws', but Frege did not say much about these (see note 2 above).

[14] See Hale 1987: 5–10, for a lucid discussion of this matter.

that 'these philosophers have so far not given a formulation of their question in terms of the common scientific language. Therefore our judgement must be that they have not succeeded in giving to the [ontological] question . . . any cognitive content. Unless and until they supply a clear cognitive interpretation, we are justified in our suspicion that their question is a pseudo-question . . .' We see here a tendency toward naturalism, common among empiricists (see ch. 1, §3 and ch. 4, §3). The idea is that science has the best, perhaps the only, line on truth and so any meaningful question must be cast in scientific terms. The ontological question is not 'theoretical' or scientific, and so it is meaningless.

What of the trivial, affirmative answer, deriving the existence of numbers from the proof that there are prime numbers greater than 10? Carnap delineates a distinction:

> Are there properties, classes, numbers, propositions? In order to understand more clearly the nature of these and related problems, it is above all necessary to recognize a fundamental distinction between two kinds of questions concerning the existence or reality of entities. If someone wishes to speak in his language about a new kind of entities, he has to introduce a system of new ways of speaking, subject to new rules; we shall call this procedure the construction of a linguistic *framework* for the new entities in question. And now we must distinguish two kinds of questions of existence: first, questions of existence of certain entities of the new kind *within the framework*; we call them *internal questions*; and second, questions concerning the existence or reality *of the system of entities as a whole*, called *external questions*. Internal questions and possible answers to them are formulated with the help of new forms of expressions. The answers may be found either by purely logical methods or by empirical methods, depending upon whether the framework is a logical or a factual one. An external question is of a problematic character in need of closer examination. (Carnap 1950: §2)

A 'linguistic framework' is an attempt formally to delineate a part of discourse. The framework should contain a precise grammar, indicating which expressions are legitimate sentences in the framework, and it should contain rules for the use of the sentences. Some of the rules may be empirical, indicating, for example, that one can assert such and such a sentence when one has a certain kind of experience. Other rules will be logical, indicating what inferences are allowed and which sentences can be asserted no matter what experience one has. Carnap calls the latter *analytic* truths.

Elsewhere, Carnap presented a logicist system much like Russell's (see, for example, Carnap 1931), but with one important difference. Russell took his task to be a *philosophical* analysis of the nature of propositions, concepts, classes, and numbers (see Goldfarb 1989), and so he insisted on the vicious circle principle, and thus rejected impredicative definitions. As seen above, the result was an unwieldy ramified type theory, with the ad hoc axiom of reducibility. Carnap, on the other hand, regarded his system as a linguistic framework—one among many. In developing a framework, one is free to *stipulate* the rules of the system, the only requirement being that the rules are clear and explicit. Carnap thus preferred Ramsey's impredicative simple theory of types, avoiding the principle of reducibility altogether (see note 11 above).

Carnap (1950) briefly sketches a linguistic framework called 'the system of numbers'. Its grammar includes numerals, variables, quantifiers such as 'there is a number *x* such that . . .', and signs for the arithmetic operations. Carnap indicates that this framework contains 'the customary deductive rules' for arithmetic. This framework seems to be a formal deductive system, like those developed in mathematical logic.

Define a *number framework* to be a system like Carnap's earlier logicist system or his later 'system of numbers'. With regard to any such system, there are, first of all, 'internal questions, e.g., "Is there a prime number greater than a hundred?" . . . [T]he answers are found, not by empirical investigations based on observations, but by logical analysis based on the rules for the new expressions. Therefore, the answers are here analytic, i.e., logically true' (Carnap 1950: §2). The existence of a prime greater than a hundred is an easy, straightforward consequence of the rules and definitions of the given number framework. The existence of *numbers* is an utterly trivial consequence of those rules and definitions. It follows from the stipulations that 1 is a number. 'Therefore, nobody who meant the question "Are there numbers?" in the internal sense would either assert or seriously consider a negative answer'.

Once again, Carnap held that the external question concerning the reality of the numbers is meaningless. The closest thing to a legitimate question is the advisability of adopting a given number framework, but this is a *pragmatic* matter, not calling for an absolute 'yes' or 'no' answer. We—the members of the intellectual/scientific community—are free to choose to adopt a framework, or not, based on how it furthers the goals we take on. The overall goal of the scientific enterprise is to describe and predict experience, and to control the physical world. Mathematics seems to be part of this scientific enterprise. The pragmatic question is whether one of Carnap's number frameworks serves the purposes of science better or worse than other frameworks, such as Russell's ramified type theory.

Carnap adopted and defended a principle of tolerance. Let a thousand flowers try to bloom, even if not all of them do:

> The acceptance or rejection of . . . linguistic forms in any branch of science, will finally be decided by their efficiency as instruments, the ratio of the results achieved to the amount of effort and complexity of the efforts required . . . Let us grant to those who work in any special field of investigation the freedom to use any form of expression which seems useful to them; the work in the field will sooner or later lead to the elimination of those forms which have no useful function. *Let us be cautious in making assertions and critical in examining them, but tolerant in permitting linguistic forms.* (Carnap 1950: §5)

In chapter 1, §2 above, we saw that Gödel defended impredicative definitions on the grounds of ontological realism. So did Ramsey (see note 11 above). From that perspective, an impredicative definition is a description of an existing entity with reference to other existing entities. But this requires a positive answer to the original *external* question about the existence of numbers, and so it goes by way of metaphysics. According to Gödel and Ramsey, impredicative definitions are acceptable *because* numbers and classes have an independent existence. In contrast, Carnap defended impredicative definitions on pragmatic grounds. His number framework is far more convenient than ramified type theory for the scientific purposes at hand. No further justification is required, or even coherent. Delving into the metaphysical status of properties, concepts, or numbers produces only pseudo-questions.

Unlike Mill, Carnap and the other logical positivists held that the truths of mathematics are not determined by experience. Mathematical truths are a priori, holding no matter what experience we may have. As empiricists, however, they held that every factual matter must ultimately be decided by experience. So the logical positivists concluded that mathematical truths have no factual content. For Carnap, the truths about the natural numbers may be called 'framework principles' since they emerge from the rules for using a number framework.

A later member of the school, Alfred J. Ayer (1946: ch. 4), put it clearly:

> For whereas a scientific generalization is readily admitted to be fallible, the truths of mathematics and logic appear to everyone to be necessary and certain. But if empiricism is correct no proposition which has a factual content can be necessary or certain. Accordingly the empiricist must deal with the truths of mathematics and logic in one of the two following ways: he must say either that they are not necessary truths . . . or he must say that they have no factual content, and then he must explain how a proposition which is empty of all factual content can be true and useful and surprising.

Ayer wrote that, contra Mill, mathematical truths are necessary, but he added that they do not say anything about the way the world is. We 'cannot abandon [the truths of logic and mathematics] without contradicting ourselves, without sinning against the rules which govern the use of language'. For Carnap, the 'rules which govern the use of language' are found in the various linguistic frameworks.

The logical positivists thus eliminated the very possibility of synthetic propositions that are knowable a priori. As Ayer put it, a proposition is synthetic, or has factual content, only if its truth or falsehood 'is determined by the facts of experience'. A proposition is analytic 'when its validity depends solely on the definitions of the symbols it contains'. For Ayer, this exhausts the cases. He added that although analytic propositions 'give us no information about any empirical situation, they do enlighten us by illustrating the way in which we use certain symbols'.

The logical positivists brought geometry into the fold. The axioms of, say, Euclidean geometry are 'simply definitions' of primitive terms like 'point' and 'line'. Ayer wrote: 'if what appears to be a Euclidean triangle is found by measurement not to have angles totaling 180 degrees, we do not say we have met with an instance which invalidates the mathematical proposition that the sum of the three angles of a Euclidean triangle is 180 degrees. We say that we have measured wrongly, or, more probably, that the triangle we have been measuring is not Euclidean.' Euclidean geometry, construed as a theory of pure mathematics, is a linguistic framework *à la* Carnap. The indicated theorem about the angles in a triangle is a framework principle, and so is analytic, knowable a priori. It is true by definition. There is a separate *pragmatic* or scientific issue concerning the advisability of adopting this framework, rather than one of the non-Euclidean geometries, for physics. This last is not a *mathematical* question.

In addition to Carnap and Ayer, the major logical positivists included the other members of the so-called 'Vienna Circle', such as Moritz Schlick, Gustav Bergmann, Herbert Feigl, Otto Neurath, and Friedrich Waismann. Outside Vienna, there is C. W. Morris, and Ernest Nagel. That movement had pretty much run its course by the 1960s, if not before, but the position on mathematics was not the main reason for the decline of logical positivism. Logical positivism shared the problem with traditional (radical) empiricism of describing the basis of knowledge. Can we distinguish observation from theory, and can we sharply distinguish mathematics from the rest of scientific theory? The success of mathematical logic led the positivists to attempt a logic of confirmation, that would relate empirical observation to scientific and mathematical theory. Yet no compelling confirmation-logic was forthcoming. These failures led to difficulty in formulating the central thesis that every factual (non-analytic) statement is verifiable. What exactly is it to be verifiable? The verifiability thesis proved untenable, even on ever-weaker notions of verification.

Some critics pointed out that the very statement of logical positivism undermines the view. Consider, for example, the proposition that every meaningful statement is either analytic or verifiable (in some sense) through experience. Apparently, *this* proposition is not analytic, in the sense of being true in virtue of the meaning of the words it contains. Also, the proposition does not seem subject to verification by experience, in any sense of the term. Thus, logical positivism seems to brand itself as a banned metaphysical doctrine. Many of Carnap's own philosophical statements, needed to outline the programme, do not seem to be made *within* a fixed linguistic framework. Indeed, his statements are *about* linguistic frameworks, and so are 'external' to any given framework. Does this turn Carnap's own work into meaningless 'pseudo-statements'?

One influential attack against logical positivism came from Carnap's most influential student, Quine, who argued that there is no distinction between analytic and synthetic statements, or at least no distinction that serves the purposes of logical positivism. According to Quine, there is no sharp distinction between the role of language and the role of the world in determining the truth or falsehood of meaningful statements. Quine proposed a holistic approach to scientific language, with observation, theory, and mathematical statements inextricably linked to each other. He shared the basic empiricist idea that observation is the basis of all knowledge, and so Quine shared a basic mistrust of much traditional metaphysics. He developed a naturalism and an empiricism closer to that of Mill in important ways. Mathematical truths are true in the same way that scientific truths and reports of observation are true. These truths are not necessary and not known a priori. We return to Quine in chapter 8, §2.

The thesis that mathematical propositions are true or false by virtue of the meaning of mathematical terminology cannot be fully adjudicated without an extended discussion of what 'meaning' is. Note, however, that one main promise of the thesis is an account of how mathematics is *known*. According to the logical positivists, knowledge of the correct use of mathematical language is sufficient for knowledge of mathematical propositions, such as the induction axiom, the prime number theorem, and even Fermat's last theorem. For Carnap, once we learn the rules of a given linguistic framework, such as the number framework or Euclidean geometry, we have everything we need for knowledge of the requisite mathematical propositions. This suggests that, epistemically, mathematical propositions can be sharply divided into self-contained groups. Each proposition p is associated with its framework P. Knowledge of the rules of P is just about all there is to knowledge of the truth or falsehood of p.

Developments in mathematics, including some results in mathematical logic, cast doubt on this promising epistemic thesis. Gödel's incompleteness theorem is that if D is an effective deductive system that contains a certain amount of arithmetic, there are sentences in the language of D which are not decided by the rules of D (see, for example, Boolos and Jeffrey 1989: ch. 15). The truth-values of many of these sentences are decided by embedding the natural numbers in a richer structure, such as the real numbers or the set-theoretic hierarchy. That is, some statements in the language of arithmetic are not knowable on the basis of the rules of a natural number framework alone. The situation is typical in mathematics. Suppose, for example, that a mathematician is interested in a certain mathematical statement *s* about a certain structure *S*. According to Carnap, if *s* is true (of *S*), then *s* is analytic and owes its truth to the linguistic framework of the structure *S*. However, the mathematician will commonly invoke structures far richer than *S* in order to prove or refute *s*. That is, the mathematician considers structures richer than *S* in order to determine the properties of *S*. No rich mathematical theory is as self-contained as Carnap's (mathematical) linguistic frameworks are supposed to be.[15]

The recent proof of Fermat's last theorem is a case in point. Anyone with a basic understanding of the terms can *understand* the statement that for any natural numbers

$$a > 0,\ b > 0,\ c > 0,\ n > 2,\ a^n + b^n \neq c^n.$$

Yet the proof eludes the grasp of all but the most sophisticated mathematicians, since it invokes concepts and structures far beyond that of the natural numbers. In this case at least, someone can understand the meanings of terms like 'natural number', 'successor function', 'addition', 'multiplication', and 'exponentiation' without having the wherewithal to see that Fermat's last theorem is true. There may be a 'self-contained' proof of this theorem—that is, a proof that does not go beyond the properties of the natural numbers. Perhaps Fermat himself discovered such a proof, but contemporary mathematicians did not learn (and do not know) the theorem via that route. Nevertheless, the theorem is clearly about the natural numbers.

One retreat would be for the logical positivist to concede that only some truths of, say, arithmetic, are analytic, or otherwise determined by the meanings of arithmetic terminology. Perhaps someone can maintain that a basic core of arithmetic truths are analytic. What of the other, non-core propositions? Are those synthetic? If so, are they somehow verifiable in observation?

Another option would be for the logical positivist to maintain the thesis that mathematical statements are true by virtue of their meaning, and concede that one can have the knowledge necessary to understand a given true proposition without thereby having the

[15] In the next chapter we will see this 'incompleteness' phenomenon undermine another once prominent philosophy of mathematics, formalism.

resources to know that it is true. The idea is that when we embed the natural numbers in a richer structure, we can thereby learn more about what *follows from the meaning* of the original mathematical terminology. The logical positivist thus needs a rich and open-ended notion of logical consequence and he needs to explicate this notion of consequence before claiming an understanding of mathematical knowledge. Until this notion of consequence is supplied and evaluated, it is not clear how much progress one can claim on the epistemology of mathematics.

4. Contemporary Views

Variations of Frege's approach to mathematics are vigorously pursued today, in the work of Crispin Wright, beginning with *Frege's Conception of Numbers as Objects* (1983), and others like Bob Hale (1987) and Neil Tennant (1997). Define a *neo-logicist* to be someone who maintains the following two theses: (1) A significant core of mathematical truths are knowable a priori, by derivation from rules which are (all but) analytic or meaning-constitutive; and (2) this mathematics concerns an ideal realm of objects which are objective, or mind-independent in some sense.[16] This combination of views is attractive to those sympathetic to the traditional view of mathematics as a body of a priori, objective truths but worried about the standard epistemological problems faced by realism in ontology. How can we know anything about a realm of causally inert, abstract objects? The neo-logicist answers: by virtue of our knowledge of what we mean when we use mathematical language—and so she attempts to resolve the problems found in traditional logicism. The neo-logicist is probably the closest contemporary heir of Coffa's 'semantic tradition'.

Recall that two concepts *F*, *G*, are equinumerous if there is a one-to-one correspondence between the objects falling under *F* and the objects falling under G. For example, if no red cards have been issued in a soccer match, the players on one team are equinumerous with the players on the other team. Frege showed how to

[16] Frege himself clearly held the second of these theses. See note 13 above on the extent to which Frege held something like the first thesis.

define equinumerosity using logical resources without explicitly presupposing the natural numbers. Recall his (1884: §63) formulation of the thesis now known as 'Hume's principle':

> For any concepts *F*, *G*, the number of *F* is identical to the number of *G* if and only if *F* and *G* are equinumerous.

The neo-logicist programme is to bypass Frege's treatment of extensions and to work with Hume's principle, or something like it, directly. A number of authors, including Wright, have pointed out that Frege's development of arithmetic (1884, 1893) contains the essentials of a derivation of the standard axioms of arithmetic from Hume's principle (in so-called 'second-order logic'—see Shapiro 1991). Moreover, Hume's principle is consistent if (second-order) arithmetic is. In the presentation of arithmetic, Frege's only substantial use of extensions, and the ill-fated Basic Law V, was to derive Hume's principle. See, for example, Parsons 1965, Wright 1983, Hodes 1984, and Boolos 1987.

As noted above, the derivation of arithmetic from Hume's principle is now called *Frege's theorem*. No one doubts that it is a substantial mathematical achievement, illuminating the natural numbers and their foundation. The neo-logicist argues that Frege's theorem supports the aforementioned philosophical theses concerning the natural numbers.

The principle idea is that the right-hand side of Hume's principle gives the truth conditions for the left-hand side, but the left-hand side has the proper grammatical and logical form. In particular, locutions like 'the number of *F*' are genuine singular terms, the grammatical forms used to denote objects. At least some instances of the right-hand side of Hume's principle are true, on logical grounds alone. For example, it is a logical truth that the concept of 'not identical to itself' is equinumerous with the concept 'not identical to itself'. Thus, from Hume's principle, we conclude that the number of non-self-identical things is identical to the number of non-self-identical things. Letting '0' denote the number of non-self-identical things, we conclude that $0 = 0$ and so zero exists.

Following Frege, the neo-logicist then defines the number 1 to be the number of the concept 'identical to zero', defines the number 2 as the number of the concept 'either identical to zero or identical to one', and on from there in Fregean fashion. It follows from Hume's principle that these natural numbers are different from each other, and so Hume's principle cannot be satisfied in a finite domain.

Like Frege's own development, the neo-logicist requires that Hume's principle be *impredicative*, in the sense that the variable F in the locution 'the number of F' can be instantiated with concepts that themselves are defined in terms of numbers. Without this feature, the very definition of the individual numbers would fail, along with the derivation of the basic arithmetic axioms from Hume's principle. This impredicativity is consonant with the ontological realism shared by Frege and his neo-logicist followers (see Wright 1998).

Wright and Hale stop short of claiming that Hume's principle is a logical truth or, in Frege's terms, a 'general logical law'. Hume's principle is not true in virtue of its form, nor does it seem to be derivable from accepted logical laws. Wright and Hale also do not claim that Hume's principle is a *definition* of cardinal number. It is generally agreed that a definition of a term must be *eliminable* in the sense that any formula containing the defined term is equivalent to a formula not containing it. It follows from Hume's principle that there is something that is the number of non-self-identical things, in symbols $\exists x(x = 0)$. Hume's principle does not provide for an equivalent sentence lacking the number terminology. A successful definition should also be *non-creative* in the sense that it has no consequences for the rest of the language and theory. Hume's principle does have such consequences, since it entails that the universe is infinite. So Hume's principle is neither eliminative nor non-creative.

Thus, Wright and Hale do not defend the traditional logicist thesis that arithmetic truth is a species of logical truth, or that each arithmetic truth follows from general logical laws and definitions. Hence the 'neo' in 'neo-logicism'. However, they argue that Hume's principle is 'analytic of' the concept of *natural number*. Thus, the programme preserves the necessity of at least the basic arithmetic truths and it shows how these truths can be known a priori. In a later work, Wright (1997: 210–11) wrote:

Frege's theorem will . . . ensure . . . that the fundamental laws of arithmetic can be derived within a system of second-order logic augmented by a principle whose role is to *explain*, if not exactly to define, the general notion of identity of cardinal number, and that this explanation proceeds in terms of a notion which can be defined in terms of second-order logic. If such an explanatory principle . . . can be regarded as *analytic*, then that should suffice . . . to demonstrate the analyticity of arithmetic. Even if that term is found troubling, . . . it will remain that Hume's principle—like any principle serving implicitly to define a certain concept—will be available without significant epistemological presupposition . . . So one clear a priori route into a recognition of the truth of . . . the fundamental laws of arithmetic . . . will have been made out. And if in addition [Hume's principle] may be viewed as a *complete* explanation—as showing how the concept of cardinal number may be fully understood on a purely logical basis—then arithmetic will have been shown up by Hume's principle . . . as transcending logic only to the extent that it makes use of a *logical* abstraction principle—one [that] deploys only logical notions. So . . . there will be an a priori route from a mastery of second-order logic to a full understanding and grasp of the truth of the fundamental laws of arithmetic. Such an epistemological route . . . would be an outcome still worth describing as logicism . . .

The key claim here is that Hume's principle does not have significant epistemological presuppositions. It is essential to the project that when attempting to establish a basic arithmetic truth, we need not invoke Kantian intuition, empirical fruitfulness, and so on.

Like the original Fregean logicism, the neo-logicist programme has a chance at success only if second-order logic is in fact logic. If substantial mathematics is already built into the logic, then as far as traditional logicism goes, Frege's theorem begs the question. What matters for neo-logicism is whether the axioms and rules of second-order logic are *analytic*, or meaning-constitutive in the requisite sense, or are otherwise free of substantial epistemological presuppositions. The status of second-order logic is an ongoing issue in contemporary philosophy. Quine (1986: ch. 5), for example, claims that second-order logic is set-theory in disguise, a 'wolf in sheep's clothing'. For a sample of the debate, see Boolos 1975, 1984, Tharp 1975, Wagner 1987, and Shapiro 1991. This reiterates the point at the end of the previous section that the underlying logical principles must be made explicit and their epistemic status clearly delineated before one can claim the virtues of a logicist programme. Lacking an examination of the logic, it is not clear what has been accomplished.

As we saw, Frege himself demurred from taking Hume's principle as the ultimate foundation for arithmetic because Hume's principle only determines identities of the form 'the number of F = the number of G'. That is, Hume's principle does not determine the truth value of sentences in the form 'the number of F = t', where t is an arbitrary singular term. The neo-logicist does not adopt Frege's resolution involving extensions, nor does he follow Russell in rejecting the existence of numbers (nor Carnap in rejecting the question of existence). Thus, the 'Caesar problem' is an active and open issue on the neo-logicist agenda. That is, the neo-logicist seeks to do what Hume's principle alone does not, to settle the identity between terms denoting natural numbers and other singular terms (see Hale 1994 and Sullivan and Potter 1997).

Hume's principle is an *abstraction*—from the relation of equinumerosity to statements about numbers. It is one of a genus of abstraction principles of the form:

$$\textbf{(ABS)} \qquad @\alpha = @\beta \text{ if and only if } E(\alpha,\beta),$$

where $E(\alpha,\beta)$ is a special kind of relation, called an 'equivalence', and '@' is a new function symbol, so that '$@\alpha$' and '$@\beta$' are singular terms.[17] Frege invokes two other abstraction principles, both in the form (ABS). One is at least relatively innocuous: the direction of l is identical to the direction of l' if and only if l is parallel to l'. The other example is his infamous, and inconsistent, Basic Law V:

> For any concepts F, G, the extension of F is identical to the extension of G if and only if for every object a, Fa if and only if Ga,

introduced as part of the theory of extensions.

The neo-logicist programme depends on the legitimacy of at least some abstraction principles. Wright concedes that his own proposals hinge on the proviso that 'concept-formation by abstraction' be accepted. George Boolos (e.g., 1997) argued against 'concept-formation by abstraction' as a legitimate manoeuvre for a prospective logicist. The most prevalent of his arguments is the 'bad company objection'. Boolos proposes that there is no non-ad hoc way to distinguish good abstraction principles like Hume's

[17] The relation E is an equivalence if (1) for every α, $E(\alpha,\alpha)$ (reflexivity), (2) for every α,β, if $E(\alpha,\beta)$ then $E(\beta,\alpha)$ (symmetry), and (3) for every α,β,γ, if $E(\alpha,\beta)$ and $E(\beta,\gamma)$, then $E(\alpha,\gamma)$ (transitivity).

principle, from bad ones like Basic Law V. To be sure, Hume's principle is consistent while Basic Law V is not, but that distinction is too coarse-grained. Hume's principle is an 'axiom of infinity' in the sense that it is satisfiable only in infinite domains. Boolos points out that there are consistent abstraction principles, with the same form (ABS) as Hume's principle (and Basic Law V) that are satisfiable only in *finite* domains. If Hume's principle is acceptable, then so are these others. However, the finite principles are incompatible with Hume's principle. How then to distinguish the legitimate abstraction principles? Wright's (1997) response is to delimit and defend certain conservation principles which rule out the bad abstraction principles and allow the good ones, Hume's principle in particular. The debate continues, but perhaps with less intensity after Boolos's tragic death in 1996.

The neo-logicist project, as developed thus far, only applies to the natural numbers and basic arithmetic. As significant as this may be, arithmetic is only a small part of mathematics. Another major item on the neo-logicist agenda is to extend the treatment to cover other areas of mathematics, like real analysis, functional analysis, and perhaps geometry and set theory. The programme involves the search for abstraction principles rich enough to characterize more powerful mathematical theories. See Wright 1997: 233–44 and Hale 2000 for attempts in this direction.

In sum, then, logicism is not dead. It is an active and potentially fruitful ongoing research programme in the philosophy of mathematics.

5. Further Reading

Many of the primary sources cited above are readable and readily available. Frege 1884 has been translated into English (by J. L. Austin), and Russell 1919 was republished in 1993 as a Dover paperback. Ayer 1946 remains a classic work. The Benacerraf and Putnam 1983 anthology contains much of the original material on logicism (in English translation if necessary), including Carnap 1931 and 1950, and selections from Frege 1884 (with different translation), Russell 1919, and Ayer 1946 (and a related piece, Hempel 1945). Resnik 1980 and Dummett 1991 are lucid, important secondary sources on Fregean logicism. See also the papers collected in Demopoulos 1995 and the second part of Boolos 1998. Several of the papers in Heck 1997 deal with neo-logicism, and the topic frequently appears in *Philosophia Mathematica.* For different logicist approaches see Dedekind 1872, 1888 (published together in translation as a Dover paperback) and Hodes 1984.